

# Local random quantum circuits: ensemble CP maps and swap algebras

Paolo Zanardi

*Department of Physics and Astronomy, and Center for Quantum Information Science & Technology,  
University of Southern California, Los Angeles, CA 90089-0484 and  
Centre for Quantum Technologies, National University of Singapore, 2 Science Drive 3, Singapore 117542*

We define different classes of local random quantum circuits (L-RQC) and show that: a) statistical properties of L-RQC are encoded into an associated family of completely positive maps and b) average purity dynamics can be described by the action of these maps on operator algebras of permutations (swap algebras). An exactly solvable one-dimensional case is analyzed to illustrate the power of the swap algebra formalism. More in general, we prove short time area-law bounds on average purity for uncorrelated L-RQC and infinite time results for both the uncorrelated and correlated cases.

## I. INTRODUCTION

The quantum-mechanical state space of a many-body system (say  $N$  particles) is huge, as its dimension scales exponentially with  $N$ . The exact specification of any quantum state therefore requires, its face value, an exponential number of parameters, a daunting task both numerically and experimentally. As in classical statistical mechanics one is then naturally led to resort to *probabilistic methods* to explore the typical properties of *ensembles* of quantum states. The question now becomes: what are the “natural” probability distribution laws one should consider over the state space? A first conceptually and mathematically appealing answer is: the group-theoretic Haar distribution [1]. This latter choice in fact seems to implement the principle of minimal a priori assumptions and at the same time allows one to exploit the powerful computational tools of representation theory [1]. On the other hand the Haar measure presents problems from a quantum information theoretic perspective. In fact it has been known for almost a decade now that sampling *exactly* the Haar distribution over a many-body Hilbert space (say  $N$  particles) by a quantum circuit made out of local gates i.e., a quantum computer, is an exponentially (in  $N$ ) hard task [2]. This result has been recently strengthened by considering all possible states that are achievable by time-dependent local Hamiltonians acting for a polynomial amount of time: the quantum states that are produced in this way occupy an exponentially small volume in Hilbert space [3]. This implies that the overwhelming majority of states in Hilbert space are not physical as they can only be produced after an exponentially long time. This startling fact led the authors of [3] to dub the Hilbert space “a convenient illusion” (sic).

However, generic properties of Haar distributed quantum states play a crucial role in the recent revival of interest on rigorous foundations of statistical mechanics started with the seminal papers [4] and [5]. In the light of the above mentioned findings about the unphysical nature of Haar distributed random states, this foundation for statistical ensembles appears to be conceptually questionable. How does one introduce a family of “physically accessible” states endowed with a natural probability measure? In Ref. [6] a first attempt to undertake this endeavor was made by introducing a family of random matrix product states, in this paper we will focus primarily on *local random quantum circuits* (L-RQC) families [7–13]. Roughly speaking (a precise mathematical definition will be given in the next section) a L-RQC is a random sequence of  $k$  random unitary transformations (quantum gates) each of which acts on small number of localized quantum degrees of freedom. The states of  $N$  particles obtained by the action of a L-RQC (with  $k = \text{Poly}(N)$ ) on an initial e.g., factorized, fiducial one are now regarded as the physically accessible ones.

A second important physical motivation for the introduction of L-RQC comes from the perspective of quantum control and quantum information processing with limited resources. In this case lack of ideal experimental ability may prevent one to know the precise structure of the quantum gate one has enacted as well, in the case of bounded spatial resolution, the precise set of local degrees of freedom e.g., qubits involved. This scenario naturally leads to a modelization in which both unitary transformations and their supports are effectively random variables.

A third compelling physical motivation behind the introduction of L-RQCs is that they provide a natural way to “simulate” the quantum dynamics of local random (time-dependent) Hamiltonians with a discrete circuit. This, for example, can be seen by expanding the evolution generated by a local-Hamiltonian using a Suzuki-Trotter type of factorization[3]. In this way the study of the ensembles for circuits whose length increase unboundedly i.e., the limit  $k \rightarrow \infty$ , will shed light on the fundamental problem of *quantum equilibration* [14]. Finally, L-RQC have been recently considered in the context of  $t$ -designs theory [7, 8]. Even if this latter one is a quite important area of application of L-RQCs,  $t$ -design theory is not the focus of the present work; it will briefly touch upon just in the last section.

The goal of this paper is to provide a mathematically precise account of the physical approach to L-RQC outlined in Refs. [9, 10]. Generalizations of the results contained therein and rigorous proofs of a few conjectures are presented.

## II. LOCAL RANDOM QUANTUM CIRCUITS

Let us start by laying down the mathematical framework we are going to elaborate on. Let  $\{h_i\}_{i \in \Lambda}$  be a collection of “local” Hilbert spaces labeled by a set  $\Lambda$  with finite cardinality  $|\Lambda|$ . For each subset  $\Omega \subset \Lambda$  we have an affiliated Hilbert space  $\mathcal{H}_\Omega := \otimes_{i \in \Omega} h_i$ , as well as the operator algebra  $\mathcal{A}(\Omega) := \mathcal{B}(\mathcal{H}_\Omega)$ . If  $\Omega_1 \subset \Omega_2$  we will consider  $\mathcal{A}(\Omega_1) \subset \mathcal{A}(\Omega_2)$  by identifying  $a_1 \in \mathcal{A}(\Omega_1)$  with  $a_2 = a_1 \otimes \mathbb{1}_{\Omega_2 - \Omega_1} \in \mathcal{A}(\Omega_2)$ . In particular  $\mathcal{A}(\Omega) \subset \mathcal{A}(\Lambda)$ ,  $(\forall \Omega \subset \Lambda)$  and  $\mathcal{A}(\Lambda) \cong \bigcup_{\Omega \subset \Lambda} \mathcal{A}(\Omega)$ . For simplicity we will further assume that each local space has the same *finite* dimension  $d$  i.e.,  $h_i \cong \mathbb{C}^d$ ,  $(\forall i \in \Lambda)$ . In this case the global state-space  $\mathcal{H}_\Lambda$  is isomorphic to  $(\mathbb{C}^d)^{\otimes |\Lambda|}$  ( $\dim \mathcal{H}_\Lambda = d^{|\Lambda|}$ ).

### Definition 1

A *local random quantum circuit family*  $\mathbf{C}_k[\Xi, \mathcal{L}, q^{(k)}]$  of length  $k$  is defined by the following data:

- i) A family of probability densities  $\Xi := \{d\mu_\Omega\}_{\Omega \subset \Lambda}$  over the groups  $\mathcal{U}(\Omega) := \{U \in \mathcal{A}(\Omega) / U^\dagger = U^{-1}\}$
- ii) A subset  $\mathcal{L}$  of the power set  $2^\Lambda$ . Elements of  $\mathcal{L}$  will be referred to as the *local regions*.
- ii) A probability law  $q^{(k)} : \mathcal{L}^k \rightarrow [0, 1] / \Omega := (\Omega_1, \dots, \Omega_k) \rightarrow q^{(k)}(\Omega)$ ,  $\sum_{\Omega \in \mathcal{L}^k} q^{(k)}(\Omega) = 1$ .

A local quantum circuit of length  $k$  is a unitary in  $\mathcal{U}(\Lambda)$  of the form  $U_\Omega := U_k U_{k-1} \dots U_1$  where  $\forall i \in \{1, \dots, k\} \Rightarrow a) U_i \in \mathcal{U}(\Omega_i)$ ; b)  $\Omega \in \mathcal{L}^k$ .

Roughly speaking, the idea is that the  $U_\Omega$ ’s are unitary-valued random variables distributed according the law  $q^{(k)}(\Omega) dU_\Omega$  ( $dU_\Omega = \prod_{i=1}^k d\mu_{\Omega_i}(U_{\Omega_i})$ ). Two different layers of randomness are involved in our construction of L-RQC, they are associated with items i) and ii) in Def. 1 respectively. The first one is a stochastic process (of length  $k$ ) in which  $k$  local regions of the base set  $\Lambda$  are selected according to the joint probability distribution  $q^{(k)}$ . The second is the selection, according to the  $d\mu_\Omega$ ’s of  $k$  unitaries acting on the state-space of each of the local regions randomly selected in the previous step.

The physical picture behind these definitions is quite simple: the experimenter has access only to random unitaries localized over regions  $\Omega$  belonging to a distinguished subset  $\mathcal{L} \subset 2^\Lambda$ . The definition of the local regions set  $\mathcal{L}$  is critical and it is where extra (physically motivated) input is needed. The idea is that the  $\Omega \in \mathcal{L}$  are “small local” subsets of  $\Lambda$ . In the following we will define RQCs over (hyper)graphs, in that case  $\Lambda$  is the vertex set and  $\mathcal{L}$  will coincide with the set of (hyper)edges; in general one may think of the local regions as sets of cardinality  $O(1)$  (where  $|\Lambda|$  is the large scaling quantity). Also, in the case in which  $\Lambda$  is naturally equipped with a metric structure e.g., the graph theoretic distance, the local regions will also be required to have small diameter i.e.,  $\text{diam}(\Omega) = O(1)$ .

In the following we will almost exclusively restrict ourselves to the case where all the  $d\mu_\Omega$ ’s are the Haar measure over  $\mathcal{U}(\Omega)$ . This symmetry assumption is crucial for our analysis as it allows one to resort systematically to powerful group representation theory tools [1]. Accordingly we will hereafter drop the  $\Xi$  symbol in the definition of  $\mathbf{C}_k$ . Notice however that other choices are possible. For example in [13] the  $d\mu$ ’s were concentrated on a finite set of *universal* gates. A first goal for this general approach is to identify families of L-RQC that are at the same time physically motivated and amenable to rigorous mathematical analysis. A couple of L-RQC families that stands out as a natural target for further investigations is given by the following types of  $q^{(k)}$

### Definition 2

- *Time dependent Markovian L- RQC:*

$$q^{(k)}(\Omega) = M^{(k-1)}(\Omega_k, \Omega_{k-1}) M^{(k-2)}(\Omega_{k-1}, \Omega_{k-2}) \dots M^{(1)}(\Omega_2, \Omega_1) q^{(1)}(\Omega_1)$$

where  $\{M^{(j)}\}_{j=1}^{k-1}$  are  $k-1$  stochastic  $|\mathcal{L}| \times |\mathcal{L}|$  matrices and  $q^{(1)}$  is a distribution over  $\mathcal{L}$ . If  $M^{(j)} = M$ ,  $(\forall j)$  we have a time-independent Markov process over  $\mathcal{L}$ .

- *Uncorrelated L-RQC:*  $q^{(k)}(\Omega) = \prod_{j=1}^k p^{(j)}(\Omega_j)$ , where the  $p^{(j)}$  are  $k$  distribution laws over  $\mathcal{L}$ . If  $q^{(j)} = q$ ,  $\forall j$  we have a time-independent uncorrelated process.

Even in these restricted setups the landscape of possibilities is huge and one has to resort, once again, to physical input to further specialize the models. For example if  $\Lambda$  is the vertex set of a graph  $\mathcal{G} := (\Lambda, E)$  and  $\mathcal{L} = E$  one may consider  $M^{(j)}$  such that  $M^{(j)}(\Omega_1, \Omega_2) \neq 0$  iff the edges  $\Omega_1$  and  $\Omega_2$  share at least one vertex. In this way the  $M^{(j)}$ ’s define a sort of *random walk* on the graph  $\mathcal{G}$ .

In passing we notice that one can use  $\mathbf{C}_k[\mathcal{L}, q^{(k)}]$  to turn  $\mathcal{A}(\Lambda)$  in to a *noncommutative probability space* [15] with the expectation  $\phi : \mathcal{A}(\Lambda) \rightarrow \mathbf{C} / A \rightarrow \phi(A)$ , where

$$\phi(A) := \sum_{\Omega \in \mathcal{L}^k} q^{(k)}(\Omega) \int dU_{\Omega} \langle \omega, U_{\Omega}^{\dagger} A U_{\Omega} \rangle \quad (1)$$

Here  $\langle A, B \rangle := \text{Tr}(A^{\dagger} B)$  the standard Hilbert-Schmidt scalar product over  $\mathcal{A}(\Lambda)$  and  $\omega$  is an distinguished (initial) density operator in  $\mathcal{A}(\Lambda)$ .

A physically compelling model of L-RQC that received much attention recently is based of graph structures  $\mathcal{G} = (\Lambda, E)$  [7–9, 13]. In our formalism this amount to say that i)  $\Lambda$  is the set of vertices of a graph  $\mathcal{G}$ , ii)  $\mathcal{L} = E \subset \Lambda^2$  is the set of edges of  $\mathcal{G}$ . Most of the literature on random quantum circuits so far has focused on this graph case in the time-independent uncorrelated case 2) and with  $\mathcal{G}$  being the complete graph. One-dimensional correlated cases have been discussed in [9, 10].

### III. ENSEMBLE MAPS

In this section we will introduce and start to analyze a set of completely positive (CP) maps [16] naturally affiliated with any L-RQC family [7, 12]. These maps comprise the full statistical content of the L-RQC family and play a central role in the following of the paper.

Using a compact notation we can write (1) in the symbolic form  $\phi(A) = \overline{\langle \omega, \mathbf{U}^{\dagger} A \mathbf{U} \rangle}^{\mathbf{U}}$ . For any fixed observable  $A \in \mathcal{A}(\Lambda)$  the mapping  $X_A : \mathbf{U} \rightarrow \langle \omega, \mathbf{U}^{\dagger} A \mathbf{U} \rangle$  defines a classical (commutative) random variable. The statistics of  $X_A$  describes the fluctuations of the (purely) quantum expectation  $\langle \omega, \mathbf{U}^{\dagger} A \mathbf{U} \rangle$  over the ensemble of the local random circuits  $\mathbf{U}$  in  $\mathbf{C}_k[\mathcal{L}, q^{(k)}]$ . While the  $p$ -moments of the noncommutative random variables  $A$  ( $p \in \mathbf{N}$ ) are defined by  $\mu_p(A) = \phi(A^p)$  the moments of the corresponding classical variables are given by  $\mu_p(X_A) = \overline{X_A^p}^{\mathbf{U}} = \overline{\langle \omega, \mathbf{U}^{\dagger} A \mathbf{U} \rangle^p}^{\mathbf{U}} = \langle \omega^{\otimes p}, \overline{(\mathbf{U}^{\dagger} A \mathbf{U})^{\otimes p}}^{\mathbf{U}} \rangle$ . These moments can be conveniently expressed in terms of a family of maps  $\mathcal{R}_p$  associated with  $\mathbf{C}_k[\mathcal{L}, q^{(k)}] : \mu_p(X_A) := \langle \omega^{\otimes p}, \mathcal{R}_p(A^{\otimes p}) \rangle$  [9, 12]. Here  $\mathcal{R}_p : \mathcal{A}(\Lambda)^{\otimes p} \rightarrow \mathcal{A}(\Lambda)^{\otimes p} / A \rightarrow \overline{(\mathbf{U}^{\dagger} A \mathbf{U})^{\otimes p}}^{\mathbf{U}}$  are completely positive (CP) maps [16]. More explicitly, if  $A \in \mathcal{A}(\Lambda)^{\otimes p}$ , one can write

$$\mathcal{R}_p(A) = \sum_{\Omega \in \mathcal{L}^k} q^{(k)}(\Omega) \mathcal{R}_{p,\Omega}(A), \quad \mathcal{R}_{p,\Omega}(A) := \int dU_{\Omega} (U_{\Omega}^{\dagger})^{\otimes p} A U_{\Omega}^{\otimes p} \quad (2)$$

We will refer to the  $\mathcal{R}_p$ 's as the *ensemble maps*. Indeed, the distribution law for any  $X_A$ 's is determined by the Fourier transform of the characteristic function  $\chi_A(t) := \sum_{p=0}^{\infty} \frac{(it)^p}{p!} \mu_p(X_A)$ . This can be in turn expressed as  $\langle \omega_{\infty}, \mathcal{R}_{\infty}(t)(A_{\infty}) \rangle$  where  $x_{\infty} := \oplus_{p=0}^{\infty} x^{\otimes p}$ , ( $x = A, \omega$ ) and  $\mathcal{R}_{\infty}(t) := \oplus_{p=0}^{\infty} \frac{(it)^p}{p!} \mathcal{R}_p$  is a formal CP-map over the *full (operator) Fock space*  $\mathcal{A}_{\infty}(\Lambda) = \oplus_{p=0}^{\infty} \mathcal{A}(\Lambda)^{\otimes p}$ .

The ensemble maps  $\mathcal{R}_p$ 's are clearly uniquely defined by the data in  $\mathbf{C}_k[\mathcal{L}, q^{(k)}]$  and should be characterized for physically relevant L-RQC families. In this paper we will often consider the  $\mathcal{R}_p$  as operators over the Hilbert-Schmidt space  $\mathcal{A}(\Omega)^{\otimes p}$ ; their norms are defined accordingly i.e.,  $\|\mathcal{R}_p\| := \sup\{\|\mathcal{R}_p(A)\| / \|A\| := \sqrt{\text{Tr} A^{\dagger} A} = 1, A \in \mathcal{A}(\Omega)\}$ . Here below we give a summary of their general properties.

#### Proposition 0

- i) The maps  $\mathcal{R}_p$  are: trace-preserving ( $\text{tr} \mathcal{R}_p(X) = \text{tr}(X)$ ) and unital ( $\mathcal{R}_p(\mathbb{1}) = \mathbb{1}$ ).
- ii) If  $q^{(k)}(\Omega) = q^{(k)}(\tilde{\Omega})$ , ( $\tilde{\Omega} := (\Omega_k, \dots, \Omega_1)$ ),  $\forall \Omega$ , the  $\mathcal{R}_p$ 's are hermitean with respect to the Hilbert-Schmidt scalar product over  $\mathcal{A}(\Lambda)^{\otimes p}$ .
- iii) In the uncorrelated case (Def. 2) they factorize:  $\mathcal{R}_p = \prod_{j=1}^k \mathcal{R}_p^{(j)}$  where  $\mathcal{R}_p^{(j)}(A) = \sum_{\Omega \in \mathcal{L}} q^{(j)}(\Omega) \mathcal{R}_{p,\Omega}(A)$ ,  $\mathcal{R}_{p,\Omega}(A) := \int dU_{\Omega} (U_{\Omega}^{\dagger})^{\otimes p} A U_{\Omega}^{\otimes p}$ .
- iv) If  $dU_{\Omega}$  is the Haar measure and that the distinguished state  $\omega$  is a pure state *completely factorized over*  $\Lambda$  i.e.,  $\omega = \otimes_{i \in \Lambda} |\phi_i\rangle\langle\phi_i|$ . Then

$$\langle \omega_{\Omega}^{\otimes p}, \mathcal{R}_{p,\Omega}(A) \rangle = \langle P_p^{(+)}, A \rangle \quad (3)$$

where  $\omega_{\Omega} = \otimes_{i \in \Omega} |\phi_i\rangle\langle\phi_i|$  and  $P_p^{(+)} = \frac{(d^{|\Omega|}-1)!}{(d^{|\Omega|}+p-1)!} \sum_{\sigma \in \mathcal{S}_p} \sigma$  is the (normalized) projector onto the totally  $\mathcal{S}_p$ -symmetric subspace of  $\mathcal{H}_{\Omega}^{\otimes p}$ .

v) In the Haar measure case the  $\mathcal{R}_p^{(j)}$  are positive semi-definite operator ( $\langle X, \mathcal{R}_p^{(j)}(X) \rangle \geq 0, \forall X$ ), their spectra are contained  $[0, 1]$  and always contain 1.

**Proof.**— i)–iii) follow easily from Eq (2).

iv) In this case representation theory implies that the  $\mathcal{R}_{p,\Omega}$ 's (in 2,  $k = 1$ ) are *projections* on the commutant of the representations  $U \in \mathcal{U}(\mathcal{H}_\Omega) \rightarrow U^{\otimes p} \in \mathcal{U}(\mathcal{H}_\Omega^{\otimes p})$  [1]. By the Schur-Weyl duality this commutant is the algebra generated by the permutations  $\mathcal{S}_p \ni \sigma : \mathcal{H}_\Omega^{\otimes p} \rightarrow \mathcal{H}_\Omega^{\otimes p} / \otimes_{i=1}^p \phi_i \rightarrow \otimes_{i=1}^p \phi_{\sigma(i)}$  [1]. Moreover, in the computations of the moments one has eventually to contract with  $\omega^{\otimes p}$  whose support is entirely contained in the  $\mathcal{S}_p$ -symmetric subspace of  $\mathcal{H}_\Omega^{\otimes p}$ . This implies that the only relevant part of the projection is the one associated with the identity irrep of  $\mathcal{S}_p$ . This leads to the explicit formula (3).

v) In the Haar measure case each of the  $\mathcal{R}_p^{(j)}$  in iii) of Prop. 0 is a convex combination of the projectors  $\mathcal{R}_{p,\Omega}$  and it is therefore a positive semi-definite. Moreover,  $\|\mathcal{R}_p^{(j)}\| = \|\sum_{\Omega \in \mathcal{L}} q^{(j)}(\Omega) \mathcal{R}_{p,\Omega}\| \leq \sum_{\Omega \in \mathcal{L}} q^{(j)}(\Omega) \|\mathcal{R}_{p,\Omega}\| = \sum_{\Omega \in \mathcal{L}} q^{(j)}(\Omega) = 1$ , hence  $\text{Sp}(\mathcal{R}_p^{(j)}) \subset [0, 1]$ . Since e.g.  $\mathbb{1}$  is always a fixed point  $1 \in \text{Sp}(\mathcal{R}_p^{(j)})$ .  $\square$

#### IV. FIRST MOMENTS

In this section we will analyze the L-RQC first moments of quantum observables by means of the ensemble maps  $\mathcal{R}_1$ . In particular the dynamics of these first moments can be studied in the limit in which the circuit size grows unboundedly i.e.,  $k \rightarrow \infty$ .

Let  $\omega \in \mathcal{A}(\Lambda)$  be a density matrix (i.e.,  $\omega \geq 0$ ,  $\text{Tr}(\omega) = 1$ ) and  $A \in \mathcal{A}(\Lambda)$  a quantum observable. The expectation value of the random variable  $\mathbf{U} \mapsto \langle \omega, \mathbf{U}^\dagger A \mathbf{U} \rangle$  over the ensemble of *uncorrelated, time-independent* circuits of length  $k$  is given by (see previous Sect. ) by  $\phi_k(A) := \langle \omega, \mathcal{R}_1^k(A) \rangle$ , where  $\mathcal{R}_1 = \sum_{\Omega \in \mathcal{L}} q^{(1)}(\Omega) \mathcal{R}_{1,\Omega}$ . We will also assume that  $\mathcal{L}$  is a covering of  $\Lambda$  i.e.,  $\cup_{\Omega \in \mathcal{L}} \Omega = \Lambda$ .

##### Proposition 1

i) In the Haar measure case the ensemble maps  $\mathcal{R}_{1,\Omega}$  are projections:  $\mathcal{R}_{1,\Omega}(X) = d^{-|\Omega|} \mathbb{1}_\Omega \otimes \text{Tr}_\Omega(X)$ .

ii)  $\mathcal{R}_{1,\Omega_1} \mathcal{R}_{1,\Omega_2} = \mathcal{R}_{1,\Omega_2} \mathcal{R}_{1,\Omega_1} = \mathcal{R}_{1,\Omega_1 \cup \Omega_2}$ .

iii)  $\text{Sp}(\mathcal{R}_1) \subset \{q(\underline{\alpha}) := \sum_{k=1}^{|\mathcal{L}|} q^{(1)}(\Omega_k) \alpha_k / \alpha_1, \dots, \alpha_{|\mathcal{L}|} \in \mathbf{Z}_2 = \{0, 1\}\} \ni 1$

iv) The following estimate for the convergence  $\phi_k(A) \rightarrow \phi_\infty(A) = d^{-|\Lambda|} \text{Tr} A$ , holds

$$k \gg \frac{\log(\|\omega\|_2 \|A\|_2 / \epsilon) + (|\mathcal{L}| - 1) \log 2}{\log \frac{1}{1 - q^{(1)}(\Omega_*)}} \Rightarrow |\phi_k(A) - \phi_\infty(A)| \leq \epsilon \quad (4)$$

where  $\Omega_* = \arg \min_{\Omega} q^{(1)}(\Omega)$ .

**Proof.**—

i) Follows from the Schur Lemma [1]. In particular if  $X = X_\Omega \otimes X_{\Omega^c} \in \mathcal{A}(\Omega) \otimes \mathcal{A}(\Omega^c)$  one has  $\mathcal{R}_{1,\Omega}(X) = d^{-|\Omega|} \mathbb{1}_\Omega \otimes X_{\Omega^c} \text{Tr}(X_\Omega)$ .

ii) When  $\Omega_1 \cap \Omega_2 = \emptyset$  the claim is obvious. Let us assume then  $\Omega_1 \cap \Omega_2 \neq \emptyset$  and w.l.o.g. that  $\Lambda = \Omega_1 \cup \Omega_2$ . We write  $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$  where  $\Lambda_1 := \Omega_1 - \Omega_2$ ,  $\Lambda_2 := \Omega_1 \cap \Omega_2$ , and  $\Lambda_3 := \Omega_2 - \Omega_1$ . Let us consider  $X \in \mathcal{A}(\Lambda) = \otimes_{i=1}^3 \mathcal{A}(\Lambda_i)$  of the form  $X_1 \otimes X_2 \otimes X_3$ . One has  $\mathcal{R}_{1,\Omega_1}(X) = d^{-|\Omega_1|} \mathbb{1}_{\Lambda_1} \otimes \mathbb{1}_{\Lambda_2} \otimes X_3 \text{Tr} X_1 \text{Tr} X_2$ , therefore  $\mathcal{R}_{1,\Omega_2}(\mathcal{R}_{1,\Omega_1}(X)) = d^{-|\Omega_2|} d^{-|\Omega_1|} \mathbb{1}_{\Lambda_1} \otimes \mathbb{1}_{\Lambda_2} \otimes \mathbb{1}_{\Lambda_3} \text{Tr} \mathbb{1}_{\Lambda_2} \text{Tr} X_3 \text{Tr} X_1 \text{Tr} X_2 = d^{-|\Omega_1| - |\Omega_2| + |\Omega_1 \cap \Omega_2|} \mathbb{1}_\Lambda \prod_{i=1}^3 \text{Tr} X_i = d^{-|\Lambda|} \mathbb{1}_\Lambda \text{Tr} X$ , ( $|\Lambda| = |\Omega_1 \cup \Omega_2| = |\Omega_1| + |\Omega_2| - |\Omega_1 \cap \Omega_2|$ ). Computing  $\mathcal{R}_{1,\Omega_1}(\mathcal{R}_{1,\Omega_2}(X))$  clearly gives the same result.

iii) Since  $\mathcal{R}_1$  is a convex combination of commuting projections its diagonalization is formally immediate. Let us introduce the notations  $P_k^{(0)} := 1 - \mathcal{R}_{\Omega_k}$ , and  $P_k^{(1)} := \mathcal{R}_{\Omega_k}$ , ( $\{\Omega_1, \dots, \Omega_{|\mathcal{L}|}\} = \mathcal{L}$ ). One can then decompose the identity CP map as follows  $1 = \prod_{k=1}^{|\mathcal{L}|} (\sum_{\alpha_k=0,1} P_k^{(\alpha_k)}) = \sum_{\alpha_1, \dots, \alpha_{|\mathcal{L}|}=0,1} \prod_{k=1}^{|\mathcal{L}|} P_k^{(\alpha_k)}$ . Therefore

$$\mathcal{R}_1 = \mathcal{R}_1 \cdot 1 = \sum_{\alpha_1, \dots, \alpha_{|\mathcal{L}|}=0,1} \left( \sum_{k=1}^{|\mathcal{L}|} q^{(1)}(\Omega_k) \alpha_k \right) \prod_{k=1}^{|\mathcal{L}|} P_k^{(\alpha_k)} =: \sum_{\underline{\alpha} \in \mathbf{Z}_2^{|\mathcal{L}|}} q(\underline{\alpha}) P[\underline{\alpha}] \quad (5)$$

where we used  $\mathcal{R}_{1,\Omega_k} P[\underline{\alpha}] = \alpha_k P[\underline{\alpha}] (\forall \underline{\alpha} \in \mathbf{Z}_2^{|\mathcal{L}|})$ . Notice that not all the projectors  $P[\underline{\alpha}] := \prod_{k=1}^{|\mathcal{L}|} P_k^{(\alpha_k)}$  are necessarily non zero [23], in any case  $\underline{\alpha} := (\alpha_1, \dots, \alpha_{|\mathcal{L}|}) \neq \underline{\alpha}' \Rightarrow P[\underline{\alpha}] P[\underline{\alpha}'] = 0$ . This show that the decomposition (5) is a spectral one from which the claim  $\text{Sp}(\mathcal{R}_1) \subset \{q(\underline{\alpha})\}_{\underline{\alpha} \in \mathbf{Z}_2^{|\mathcal{L}|}}$  follows.

Since  $\exists X$  such that  $P[(1, \dots, 1)](X) = \prod_{k=1}^{|\mathcal{L}|} \mathcal{R}_{1, \Omega_k}(X) = \mathcal{R}_{1, \cup_{k=1}^{|\mathcal{L}|} \Omega_k}(X) = \mathcal{R}_{1, \Lambda}(X) = d^{-|\Lambda|} \mathbb{1}_\Lambda \text{Tr} X \neq 0$  one sees that  $P[(1, \dots, 1)] \neq 0$  and therefore  $q(1, \dots, 1) = \sum_{\Omega} q^{(1)}(\Omega) = 1$  always belong to the spectrum of  $\mathcal{R}_1$ . Moreover (Prop. 0, v) 1 the maximum eigenvalue and  $q(\alpha) = 1 - \sum_{i: \alpha_i=0} q^{(1)}(\Omega_i) < 1$  if  $\alpha \neq (1, \dots, 1)$ .

**iv)** We first observe that the second largest  $q(\alpha)$  is upper bounded by  $1 - q^{(1)}(\Omega_*)$  where  $\Omega_* = \arg \min_{\Omega} q^{(1)}(\Omega)$ . Second,  $\mathcal{R}_1^k(A) = \sum_{\underline{\alpha}} q(\underline{\alpha})^k P[\underline{\alpha}](A) = P[(1, \dots, 1)](A) + \sum_{\underline{\alpha} \neq (1, \dots, 1)} q(\alpha)^k P[\underline{\alpha}](A)$  from which  $\lim_{k \rightarrow \infty} \mathcal{R}_1^k = P[1, \dots, 1]$  and

$$\begin{aligned} |\phi_k(A) - \phi_{\infty}(A)| &= |\phi_k(A) - \langle \omega, P[1, \dots, 1](A) \rangle| \leq \sum_{\underline{\alpha} \neq (1, \dots, 1)} q(\alpha)^k |\langle \omega, P[\underline{\alpha}](A) \rangle| \\ &\leq (1 - q^{(1)}(\Omega_*))^k \sum_{\underline{\alpha} \neq (1, \dots, 1)} \|P[\underline{\alpha}]\| \|\omega\|_2 \|A\|_2 \leq (1 - q^{(1)}(\Omega_*))^k 2^{|\mathcal{L}|-1} \|\omega\|_2 \|A\|_2 \leq \epsilon \end{aligned} \quad (6)$$

Here we used Cauchy-Schwarz  $|\langle \omega, P[\underline{\alpha}](A) \rangle| \leq \|\omega\|_2 \|P[\underline{\alpha}](A)\|_2 \leq \|\omega\|_2 \|P[\underline{\alpha}]\| \|A\|_2 \leq \|\omega\|_2 \|A\|_2$  as  $P[\underline{\alpha}]$  is a projector i.e., norm one. Solving the last inequality in (6) for  $k$  gives the estimate (4).  $\square$

Whereas the asymptotic value  $\phi_{\infty}(A)$  does not depend on the distribution  $q^{(1)}$  (as long as  $\cup_{\Omega \in \mathcal{L}} \Omega = \Lambda$ ) the convergence rate may well do so. In order have a faster convergence one would like to make  $q(\Omega_*)$  as large as possible i.e.,  $q(\Omega_*) = 1/|\mathcal{L}|$ . In this case a weaker form of (4) gives  $k \gg 1/q(\Omega_*) \log(\|\omega\|_2 \|A\|_2 2^{|\mathcal{L}|-1}/\epsilon) = O(|\mathcal{L}| \log(\|\omega\|_2 \|A\|_2/\epsilon) + |\mathcal{L}|^2 \log 2) \rightarrow |\phi_k(A) - \phi_{\infty}(A)| \leq \epsilon$ . In several relevant examples one has  $|\mathcal{L}| = O(|\Lambda|)$  so that (4) gives, for  $\|A\|_2 = O(1)$ , a convergence estimate *polynomial* in the system size.

Of course (4) is a *very* crude estimate that may be strongly improved yielding faster convergence rates. For example if  $A \in \mathcal{A}(\Omega)$  then  $P[\underline{\alpha}]A = 0$  for all the  $\underline{\alpha}$ 's such that  $\alpha_k = 0$  when  $\Omega_k \subset \Omega^c$ . This constraint greatly reduces the number of non vanishing terms in (6) giving rising to a better bound. In fact, If  $|\{\Omega \in \mathcal{L} / \Omega \subset \Omega^c\}| = O(|\Omega^c|)$  then  $|\mathcal{L}|$  in (4) becomes  $|\mathcal{L}| - O(|\Omega^c|) = O(|\Omega|)$ .

## V. PURITY DYNAMICS AND SWAP ALGEBRAS

In this section we will show how the ensemble maps  $\mathcal{R}_2$  allow one to study the dynamics of average purity and how this can be done in terms of dynamically closed subalgebras of permutations. Uncorrelated as well as correlated L-RQC will be considered.

Given a density matrix  $\omega \in \mathcal{A}(\Lambda)$  the reduced state  $\omega_{\Omega} \in \mathcal{A}(\Omega)$  associated with the region  $\Omega \subset \Lambda$  is given by the partial trace  $\omega_{\Omega} = \text{tr}_{\Omega^c}(\omega)$  ( $\Omega^c := \Lambda - \Omega$ ). A family of non-linear functions that plays an important role in quantum information theory is given by the  $\alpha$ -Renyi entropies:  $S_{\alpha}(\omega_{\Omega}) := -\frac{1}{\alpha-1} \log \text{Tr}(\omega_{\Omega}^{\alpha})$ , ( $\alpha \in \mathbf{R}_0^+$ ). In particular  $\lim_{\alpha \rightarrow 1} S_{\alpha}(\omega_{\Omega}) = -\text{Tr}(\omega_{\Omega} \log \omega_{\Omega})$  is the von Neumann entropy that for pure  $\omega$  quantifies *quantum entanglement* between the two regions  $\Omega$  and  $\Omega^c$  [17]. For  $\alpha = p \in \mathbf{N}$  one can write  $\text{Tr}(\omega_{\Omega}^p) = \text{Tr}(T_{\Omega}^{(p)} \omega^{\otimes p})$  where  $T_{\Omega}^{(p)} : \mathcal{H}_{\Lambda}^{\otimes p} \rightarrow \mathcal{H}_{\Lambda}^{\otimes p}$  is the cyclic permutation  $\pi : i \rightarrow i-1 \bmod p$  acting on the  $\mathcal{H}_{\Omega}$  factors of  $\mathcal{H}_{\Lambda}^{\otimes p} \cong \mathcal{H}_{\Omega}^{\otimes p} \otimes \mathcal{H}_{\Lambda-\Omega}^{\otimes p}$ . For  $p = 2$  we will call this operator the *swap* associated with  $\Omega$  and it will be denoted by  $T_{\Omega}$ , we will also write  $\mathcal{R}_{2, \Omega}$  simply as  $\mathcal{R}_{\Omega}$  and so on. Since  $\mathcal{H}_{\Omega} = \otimes_{i \in \Omega} h_i$  one has  $T_{\Omega} = \otimes_{i \in \Omega} T_i$  where the  $T_i$  act as  $\pi$  in  $h_i^{\otimes 2}$ . The case  $p = 2$  has special relevance as the 2-Renyi entropy  $S_2(\omega_{\Omega}) = -\log \text{tr}(\omega_{\Omega}^2)$  is a convenient way to quantify quantum entanglement that is receiving a constantly growing attention from the quantum information and theoretical condensed matter communities see e.g., [18]. If the initial state  $\omega$  is evolved by a RQC  $\mathbf{U}$  in  $\mathbf{C}_k[\mathcal{L}, q^{(k)}]$  the corresponding *purity* becomes a random variable whose expectation value  $P_k := \overline{\text{Tr}((\mathbf{U}^{\otimes 2})^{\dagger} T_{\Omega} \mathbf{U}^{\otimes 2} \omega^{\otimes 2})}^{\mathbf{U}}$  is given by:

$$P_k := \sum_{\Omega \in \mathcal{L}^k} q^{(k)}(\Omega) \langle \omega^{\otimes 2}, \mathcal{R}_{\Omega}(T_{\Omega}) \rangle = \langle \omega^{\otimes 2}, \mathcal{R}(T_{\Omega}) \rangle \quad (7)$$

By convexity it follows that  $\overline{S_2}^{\mathbf{U}} \geq -\log P_k$ , namely the negative of the logarithm of (7) provides a lower bound to the average entanglement generated by evolving  $\omega$  with RQC in  $\mathbf{U}$  in  $\mathbf{C}_k[\mathcal{L}, q^{(k)}]$ . Two of the key technical insights in [9] on which we would like to build upon in this paper are contained in the next two propositions:

### Proposition 2

The swaps  $T_{\Omega}$  form an abelian group  $\mathcal{T}_{\Lambda}$  of order  $2^{|\Lambda|}$  whose elements have degree two i.e.,  $T_{\Omega}^2 = \mathbb{1}$ .

$$T_{\Omega_1} T_{\Omega_2} = T_{\Omega_1 \Delta \Omega_2}, \quad \Omega_1 \Delta \Omega_2 := (\Omega_1 - \Omega_2) \cup (\Omega_2 - \Omega_1) \quad (\forall \Omega_1, \Omega_2 \subset \Lambda) \quad (8)$$

The group algebra  $\mathbf{CT}_\Lambda$  is  $2^{|\Lambda|}$ -dimensional abelian subalgebra of  $\mathbf{CT}_\Lambda \subset \mathcal{A}(\Lambda)^{\otimes 2}$  under multiplication.  $\mathbf{CT}_\Lambda$  will be referred to as the *swap algebra* of  $\Lambda$ .

**Proof.**—  $T_{\Omega_1} T_{\Omega_2} = \left( \prod_{i \in \Omega_1} T_i \right) \left( \prod_{j \in \Omega_2} T_j \right) = \left( \prod_{i \in \Omega_1 - \Omega_2} T_i \right) \left( \prod_{i \in \Omega_1 \cap \Omega_2} T_i^2 \right) \left( \prod_{i \in \Omega_2 - \Omega_1} T_i \right) = \left( \prod_{i \in (\Omega_1 - \Omega_2) \cup (\Omega_2 - \Omega_1)} T_i \right) := T_{\Omega_1 \Delta \Omega_2}$ . Where we have used  $T_i^2 = \mathbb{1}$ ,  $\forall i \in \Lambda$ . Notice also that the  $T_\Omega$ 's are linearly independent in  $\mathcal{A}(\Lambda)^{\otimes 2}$  as each of them amounts to a permutation of the product state basis of  $\mathcal{H}_\Lambda^{\otimes 2} \cong \otimes_{i \in \Lambda} h_i^{\otimes 2}$ .  $\square$

Eq. (8) shows that  $\mathcal{T}_\Lambda$  is isomorphic to the power set of  $\Lambda$  endowed with the internal operation of symmetric set difference  $\Delta$ . The isomorphism being given by  $\Omega \rightarrow T_\Omega$ . Moreover, the swap algebra  $\mathbf{CT}_\Lambda$  is isomorphic, as a vector space, to the space  $(\mathbf{C}^2)^{\otimes |\Lambda|}$  associated with  $|\Lambda|$  qubits ( $T_\Omega \cong \otimes_{j \in \Lambda} |\chi_\Omega(j)\rangle$ ,  $\chi_\Omega$  characteristic function of  $\Omega \subset \Lambda$ ). Next proposition shows that  $\mathbf{CT}_\Lambda$  is *invariant* under the  $\mathcal{R}_\Omega$ 's.

### Proposition 3

i) If  $\Omega \subset \Lambda$  let us define its boundary (in  $\mathcal{L}$ ) as  $\partial\Omega := \{\Omega' \in \mathcal{L} / \Omega' \cap \Omega \neq \emptyset \wedge \Omega' \cap \Omega^c \neq \emptyset\}$ , then

a)  $\Omega_1 \notin \partial\Omega \Rightarrow \mathcal{R}_{\Omega_1}(T_\Omega) = T_\Omega$ , otherwise b)

$$\mathcal{R}_{\Omega_1}(T_\Omega) = \alpha_+ T_{\Omega - \Omega_1} + \alpha_- T_{\Omega \cup \Omega_1}, \quad (9)$$

where  $\alpha_\pm = (c^\pm \pm c^-)/2$ ,  $c^\pm = (d^A \pm d^B)/(d^{A+B} \pm 1)$ ,  $A = |\Omega_1 - \Omega|$ ,  $B = |\Omega \cap \Omega_1|$ .

ii) The swap algebra is invariant under all the ensemble maps  $\mathcal{R}$ .

iii) Let  $P \in L(\mathbf{CT}_\Lambda)$  s.t.  $T_\Omega \mapsto T_{\Omega^c}$ , ( $\Omega \subset \Lambda$ ) then  $[\mathcal{R}_{\Omega_1}, P] = 0$ , ( $\forall \Omega_1 \subset \Lambda$ )

**Proof.**— i) a) If  $\Omega_1 \subset \Omega^c$  then the result follows trivially from  $\mathcal{R}_{\Omega_1}(\mathbb{1}_{\Omega_1}) = \mathbb{1}_{\Omega_1}$ . If  $\Omega_1 \subset \Omega$  then it follows from  $U_{\Omega_1}^{\dagger \otimes 2} T_\Omega U_{\Omega_1}^{\otimes 2} = T_{\Omega - \Omega_1} U_{\Omega_1}^{\dagger \otimes 2} T_{\Omega_1} U_{\Omega_1}^{\otimes 2} = T_{\Omega - \Omega_1} T_{\Omega_1} (T_{\Omega_1} U_{\Omega_1}^{\dagger \otimes 2} T_{\Omega_1}) U_{\Omega_1}^{\otimes 2} = T_\Omega U_{\Omega_1}^{\dagger \otimes 2} U_{\Omega_1}^{\otimes 2} = T_\Omega$ .

b) First, one has that  $\mathcal{R}_{\Omega_1}(T_\Omega) = T_{\Omega - \Omega_1} \mathcal{R}_{\Omega_1}(T_{\Omega \cap \Omega_1})$ . In order to compute  $\mathcal{R}_{\Omega_1}(T_{\Omega \cap \Omega_1})$ , notice that  $\mathcal{R}_{\Omega_1}$  (restricted on  $\mathcal{H}_{\Omega_1}^{\otimes 2}$ ) is a projection on the 2-dimensional algebra spanned by  $\mathbb{1}$  and  $T_{\Omega_1}$  [1]. A Hilbert-Schmidt orthonormal basis of this space is given by the elements

$$F_\pm := \frac{\mathbb{1} \pm T_{\Omega_1}}{\sqrt{2d^{|\Omega_1|}(d^{|\Omega_1|} \pm 1)}}.$$

Hence  $\mathcal{R}_{\Omega_1}(T_{\Omega \cap \Omega_1}) = \sum_{\alpha=\pm} F_\alpha \text{Tr}(F_\alpha T_{\Omega \cap \Omega_1})$ . Using (8) one obtains,  $\text{Tr}(F_\alpha T_{\Omega \cap \Omega_1}) \sim \text{Tr}(T_{\Omega \cap \Omega_1} + \alpha T_{\Omega_1 - \Omega}) = d^{|\Omega \cap \Omega_1|} d^{2|\Omega_1 - \Omega|} + \alpha d^{|\Omega_1 - \Omega|} d^{2|\Omega \cap \Omega_1|}$ . Now the claim follows by noticing that  $|\Omega_1| = |\Omega_1 - \Omega| + |\Omega \cap \Omega_1|$  and that, from Eq. (8), one has  $T_{\Omega - \Omega_1} T_{\Omega_1} = T_{\Omega \cup \Omega_1}$ .

ii) It is immediate by observing that  $\mathcal{R}_\Omega = \mathcal{R}_{\Omega_1} \circ \mathcal{R}_{\Omega_2} \circ \dots \circ \mathcal{R}_{\Omega_k}$  (remind that  $\Omega = (\Omega_1, \dots, \Omega_k) \in \mathcal{L}^k$ ) and using (2) and (9).

iii) First notice that (from i))  $\alpha^\pm(\Omega, \Omega_1) = \alpha^\mp(\Omega^c, \Omega_1)$ , whence  $\mathcal{R}_{\Omega_1}(PT_\Omega) = \mathcal{R}_{\Omega_1}(T_{\Omega^c}) = \alpha^+(\Omega^c, \Omega_1) T_{\Omega^c \cap \Omega_1^c} + \alpha^-(\Omega^c, \Omega_1) T_{\Omega^c \cup \Omega_1} = \alpha^-(\Omega, \Omega_1) T_{(\Omega \cup \Omega_1)^c} + \alpha^+(\Omega, \Omega_1) T_{(\Omega \cap \Omega_1^c)^c} = \alpha^-(\Omega, \Omega_1) PT_{\Omega \cup \Omega_1} + \alpha^+(\Omega, \Omega_1) PT_{\Omega \cap \Omega_1^c} = P\mathcal{R}_{\Omega_1}(T_\Omega)$ .  $\square$

Proposition 3 is an important result. It shows that (average) purity dynamics can be mapped onto a dynamical problem in a space comprising just  $|\Lambda|$  qubits. This mapping entails a dimensional reduction  $(d^2)^{|\Lambda|} \mapsto 2^{|\Lambda|}$ . Exploiting symmetries this reduction can be made, in some cases, even stronger. To start with, map  $P$  in iii) is an involution i.e.,  $P^2 = \mathbb{1}$ , and therefore  $\mathbf{CT}_\Lambda$  breaks up in two orthogonal  $(|\Lambda| - 1)$ -qubits subspaces (corresponding to eigenvalues  $\pm 1$  of  $P$ ) that are invariant under all ensemble maps. Moreover, the action of the ensemble maps on the swap algebra can be highly reducible thus lowering even further the dimensionality of the purity dynamics problem. References [9] and [10] contain several examples (see two below) in which the entanglement dynamics is approached and solved at the swap algebra level because of the dramatic decrease of computational complexity.

### A. Uncorrelated case

In the uncorrelated case the maps  $\mathcal{R}^{(j)}$  define a (discrete time) dynamical system on  $\mathbf{CT}_\Lambda$ : the element  $T$  is mapped at “time”  $k \in \mathbf{N}$  into  $T(k) := \mathcal{R}^{(k)} \circ \mathcal{R}^{(k-1)} \dots \mathcal{R}^{(1)}(T)$ . In particular in the time independent case  $T(k) = \mathcal{R}^k(T)$ . In this section we will focus on this latter case. Since the eigenvalues of  $\mathcal{R}$  are in  $[0, 1]$  (see v) in Prop. 0) the asymptotic behavior at  $k = \infty$  is controlled by the eigenspace of  $\mathcal{R}$  with eigenvalue 1 i.e.,  $\text{Fix}(\mathcal{R}) := \{X \in \mathcal{A}(\Lambda) / \mathcal{R}(X) = X\}$ . Next proposition describes the structure of this subspace and gives an explicit formula for the limit purity.

**Proposition 4**

Let us consider  $\mathcal{L} \subset 2^\Lambda$  as an *hypergraph* in  $\Lambda$  and let  $\{C_i\}_{i=1}^K$  be the family of its connected components. [24] **i)**

$$\text{Fix}(\mathcal{R}) = \bigcap_{\Omega \in \mathcal{L}} \text{Fix}(\mathcal{R}_\Omega) = \left( \bigotimes_{i=1}^K \mathbf{C}\{\mathbb{1}, T_{C_i}\} \right) \otimes \mathcal{A}((\cup_{i=1}^K C_i)^c) \quad (10)$$

**ii)** Using the notation of Eq. (7), for a totally factorized initial state  $\omega$

$$P_\infty := \lim_{k \rightarrow \infty} \langle \omega^{\otimes 2}, \mathcal{R}^k(T_\Omega) \rangle = \prod_{i=1}^K \frac{d^{|C_i| - |\Omega_i|} + d^{|\Omega_i|}}{d^{|C_i|} + 1}, \quad (\Omega_i := \Omega \cap C_i, i = 1, \dots, K) \quad (11)$$

**Proof. – i)** Let us first prove the first equality in (10). If  $X \in \text{Fix}(\mathcal{R})$  then  $\|X\| = \|\mathcal{R}(X)\| = \|\sum_{\Omega \in \mathcal{L}} q(\Omega) \mathcal{R}_\Omega(X)\| \leq \sum_{\Omega \in \mathcal{L}} q(\Omega) \|\mathcal{R}_\Omega(X)\| \leq \sum_{\Omega \in \mathcal{L}} q(\Omega) \|\mathcal{R}_\Omega\| \|X\| = (\sum_{\Omega \in \mathcal{L}} q(\Omega)) \|X\| = \|X\|$ . Here we used  $\|\mathcal{R}_\Omega\| = 1, (\forall \Omega)$  and the normalization of  $q$ . This shows that all these inequalities are actually equalities that in turns implies that,  $\|\mathcal{R}_\Omega(X)\| = \|X\|, (\forall \Omega)$ , and, since the  $\mathcal{R}_\Omega$ 's are projections,  $\mathcal{R}_\Omega(X) = X, (\forall \Omega \in \mathcal{L})$ . Therefore  $\text{Fix}(\mathcal{R}) \subset \bigcap_{\Omega \in \mathcal{L}} \text{Fix}(\mathcal{R}_\Omega)$ . The reverse inclusion is obvious as  $q$  is a probability distribution. We now move to consider the second equality in (10). We will here work at the level of the algebra  $\mathcal{A}(\Lambda)^{\otimes 2}$ , while in Appendix A the analysis is performed at the swap algebra  $\mathbf{CT}_\Omega$  level.

Let us begin by considering just two maps  $\mathcal{R}_{\Omega_1}$  and  $\mathcal{R}_{\Omega_2}$ . Since one has that  $\text{Fix}(\mathcal{R}_{\Omega_i}) = \mathbf{C}\{\mathbb{1}, T_{\Omega_i}\} \otimes \mathcal{A}(\Omega_i^c)^{\otimes 2}, (i = 1, 2)$ , it is easy to check that there are two cases: 1) If  $\Omega_1 \cap \Omega_2 = \emptyset$  then  $F = \mathbf{C}\{\mathbb{1}, T_{\Omega_1}\} \otimes \mathbf{C}\{\mathbb{1}, T_{\Omega_2}\} \otimes A$ ; and 2) If  $\Omega_1 \cap \Omega_2 \neq \emptyset$  then  $F = \mathbf{C}\{\mathbb{1}, T_{\Omega_1 \cup \Omega_2}\} \otimes A$  where  $F = \text{Fix}(\mathcal{R}_{\Omega_1}) \cap \text{Fix}(\mathcal{R}_{\Omega_2})$  and  $A := \mathcal{A}((\Omega_1 \cup \Omega_2)^c)^{\otimes 2}$ . Case 1) is obvious, we will then consider 2).

Let  $\Lambda_1, \Lambda_2$  and  $\Lambda_3$  be as in **ii)** in the proof of Prop. 1 ( $\cup_{i=1}^3 \Lambda_i = \Omega_1 \cup \Omega_2$ ). If  $X \in \text{Fix} \mathcal{R}_{\Omega_1}$  ( $X \in \text{Fix} \mathcal{R}_{\Omega_2}$ ) one can write  $X = \alpha \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes X_3 + \beta T_1 \otimes T_2 \otimes Y_3$  ( $X = \alpha' X_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 + \beta' Y_1 \otimes T_2 \otimes T_3$ ) where  $X_3$  and  $Y_3$  ( $X_1$  and  $Y_1$ ) are arbitrary operators in  $\mathcal{A}(\Lambda_3)^{\otimes 2}$  ( $\mathcal{A}(\Lambda_1)^{\otimes 2}$ ). Therefore if  $X \in F$  one must have  $\alpha \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes X_3 + \beta T_1 \otimes T_2 \otimes Y_3 = \alpha' X_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 + \beta' Y_1 \otimes T_2 \otimes T_3$ , this equation can be solved for arbitrary  $\alpha$  and  $\beta$  iff  $\mathbb{1}_1 \otimes \mathbb{1}_2 \otimes X_3 \sim X_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3$  and  $T_1 \otimes T_2 \otimes Y_3 \sim Y_1 \otimes T_2 \otimes T_3$ . This implies  $X_1 = \mathbb{1}_1$  and  $X_3 = \mathbb{1}_3$  as well as  $Y_1 = T_1$  and  $Y_3 = T_3$ . In other terms the general element  $X \in F$  must have the form  $X = \alpha \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 + \beta T_1 \otimes T_2 \otimes T_3$  namely  $X \in \mathbf{C}\{\mathbb{1}_{\Omega_1 \cup \Omega_2}, T_{\Omega_1 \cup \Omega_2}\} = \text{Fix}(\mathcal{R}_{\Omega_1 \cup \Omega_2})$ . This shows that  $F \subset \text{Fix}(\mathcal{R}_{\Omega_1 \cup \Omega_2})$ , the opposite inclusion is obvious, hence  $F = \text{Fix}(\mathcal{R}_{\Omega_1 \cup \Omega_2})$ .

The intersection over the whole  $\mathcal{L}$  required in (10) can be now performed, using 1) and 2), by organizing the hypergraph  $\mathcal{L}$  in to its connected components  $C_i, (i = 1, \dots, K)$ . Different  $C_i$ 's give rise, thanks to 1) to the different tensor factors in (10); whereas intersections  $\bigcap_{\Omega \in \mathcal{L}} \text{Fix}(\mathcal{R}_\Omega)$  give rise, thanks to 2), to  $\text{Fix}(\mathcal{R}_{C_i}) = \mathbf{C}\{\mathbb{1}, T_{C_i}\}$ , where  $C_i := \bigcup_{\Omega \in \mathcal{L}} \Omega$ . The last factor in (10) simply reflects the fact that all the elements in the algebra over the set  $(\bigcup_{i=1}^K C_i)^c$  are (trivial) fixed points for all the  $\mathcal{R}_\Omega$ 's. (Notice that often this set will be empty). This completes the proof of (10).

**ii)** Since  $\mathcal{R}$  is a non-negative operator on the Hilbert-Schmidt space  $\mathcal{A}(\Lambda)^{\otimes 2}$  with unit norm its spectrum  $\text{Sp}(\mathcal{R}) = \{\rho_\alpha\}_\alpha$  is contained in  $[0, 1]$ . Therefore  $\mathcal{R}^k(T_\Omega) = \sum_\alpha \rho_\alpha^k F_\alpha \langle F_\alpha, T_\Omega \rangle, (\mathcal{R}(F_\alpha) = \rho_\alpha F_\alpha)$  and  $\lim_{k \rightarrow \infty} \mathcal{R}^k(T_\Omega) = \sum_{\alpha: \rho_\alpha=1} F_\alpha \langle F_\alpha, T_\Omega \rangle$ . The relevant eigenoperators  $F_\alpha$  can be chosen from an orthonormal basis of the space  $\text{Fix}(\mathcal{R})$  that we characterized in i). This latter eigenspace, according to i) has dimension at least  $2^K$  but symmetry further simplifies the computation of  $P_\infty$ . Notice indeed that, for a totally factorized state  $\omega = \otimes_{i \in \Lambda} |\phi_i\rangle \langle \phi_i|$ , one has that 1)  $\langle \omega^{\otimes 2}, T_\Omega \rangle = 1, (\forall \Omega \subset \Lambda)$  and 2)  $\langle \omega^{\otimes 2}, X_1 X_2 \rangle = \langle \omega_1^{\otimes 2}, X_1 \rangle \langle \omega_2^{\otimes 2}, X_2 \rangle$  if  $X_1$  and  $X_2$  are supported on disjoint subsets  $\Omega_1$  and  $\Omega_2$  and  $\omega_i = \otimes_{j \in \Omega_i} |\phi_j\rangle \langle \phi_j|, (i = 1, 2)$ . These two facts implies in turn that, in each of the two-dimensional  $\mathbf{C}\{\mathbb{1}, T_{C_i}\}$  factors of  $\text{Fix}(\mathcal{R})$ , only the symmetric term  $\mathbb{1} + T_{C_i}$  will have give a non vanishing contribution to purity (as  $\langle \omega^{\otimes 2}, \mathbb{1} - T_{C_i} \rangle = 0$ .) Equation (11) then follows by computing the Hilbert-Schmidt scalar product of the swap  $T_\Omega$  with the normalized  $\prod_{i=1}^K (\mathbb{1} + T_{C_i})$ . From  $\Omega = (\Omega \cap (\cup_{i=1}^K C_i)^c) \cup (\Omega \cap (\cup_{i=1}^K C_i)) = (\Omega \cap (\cup_{i=1}^K C_i)^c) \cup (\cup_{i=1}^K \Omega_i)$  it follows that  $T_\Omega = \left( \prod_{i=1}^K T_{\Omega_i} \right) T_{\Omega \cap (\cup_{i=1}^K C_i)^c}$ . Since  $\mathcal{R}$  has no non-trivial action on the last swap in  $T_\Omega$  that term (if present) gives rise to an irrelevant factor one in  $P_\infty$ . The other  $K$  swaps instead give

$$P_\infty = \prod_{i=1}^K \left\langle \frac{\mathbb{1} + T_{C_i}}{\sqrt{2d^{|C_i|}(d^{|C_i|} + 1)}}, T_{\Omega_i} \right\rangle \langle \omega^{\otimes 2}, \frac{\mathbb{1} + T_{C_i}}{\sqrt{2d^{|C_i|}(d^{|C_i|} + 1)}} \rangle = \prod_{i=1}^K \left\langle \frac{\mathbb{1} + T_{C_i}}{d^{|C_i|}(d^{|C_i|} + 1)}, T_{\Omega_i} \right\rangle,$$

computing explicitly the scalar products above (use  $\text{Tr}(T_{\Omega_i}) = d^{2(|C_i| - |\Omega_i|)} d^{|\Omega_i|}, \text{Tr}(T_{C_i} T_{\Omega_i}) = \text{Tr}(T_{C_i - \Omega_i}) = d^{|C_i| - |\Omega_i|} d^{2|\Omega_i|}$ ) proves (11).  $\square$

**Remark. –** As noticed in the above for a totally factorized state  $\omega = \otimes_{i \in \Lambda} |\phi_i\rangle \langle \phi_i|$  one has that  $\langle \omega^{\otimes 2}, T_\Omega \rangle = 1, \forall \Omega$

regardless of the choice of the  $|\phi_i\rangle$ 's. It follows that in purity calculations  $\omega^{\otimes 2}$  can be replaced by the average

$$\bigotimes_{i \in \Lambda} \int dU_i U_i^{\otimes 2} |\phi_i\rangle \langle \phi_i|^{\otimes 2} (U_i^\dagger)^{\otimes 2} = \bigotimes_{i \in \Lambda} \frac{\mathbb{1} + T_i}{d(d+1)} = \frac{1}{[d(d+1)]^{|\Lambda|}} \sum_{\alpha_1, \dots, \alpha_{|\Lambda|=0,1}} T_1^{\alpha_1} \dots T_{|\Lambda|}^{\alpha_{|\Lambda|}} =: \frac{1}{d_+^{|\Lambda|}} \Pi^+$$

where  $d_+ := d(d+1)/2$  and  $\Pi^+ := 2^{-|\Lambda|} \sum_{\Omega \subset \Lambda} T_\Omega = |\mathcal{T}_\Lambda|^{-1} \sum_{T \in \mathcal{T}_\Lambda} T$ . From this last expression is clear that  $\Pi^+$  is the projector over the identity irrep of  $\mathcal{T}_\Lambda$  acting on  $\mathcal{A}(\Lambda)^{\otimes 2}$ . Moreover,  $\|\Pi^+\|_2^2 = \text{Tr } \Pi^+ = 2^{-|\Lambda|} \sum_{\Omega \subset \Lambda} \text{Tr } T_\Omega = 2^{-|\Lambda|} \sum_{\Omega \subset \Lambda} d^{|\Omega|} d^{2(|\Lambda|-|\Omega|)} = 2^{-|\Lambda|} d^{2|\Lambda|} (1+d^{-1})^{|\Lambda|} = d_+^{|\Lambda|} \Rightarrow \|\Pi^+\|_2 = d_+^{|\Lambda|/2}$ .

**Example .-** Let  $\mathcal{G} = (\Lambda, E)$  be a connected graph and the ensemble map given by:  $\mathcal{R} = |E|^{-1} \sum_{e \in E} \mathcal{R}_e$ . This is the so-called random edge model considered in [9, 10]. Here we have that  $\mathcal{L}$  coincides with  $E$  (regarded as a family of subsets  $\{v_1, v_2\} \subset \Lambda$ ) therefore one has  $K = 1$  and  $|C_1| = |\Lambda|$  and Eq (11) becomes  $P_\infty = \frac{d^{|\Omega|} + d^{|\Omega^c|}}{d^{|\Lambda|} + 1}$ .

### B. Area Laws for the uncorrelated case

In this section we exploit the locality structure of the ensemble maps and the swap algebra relation (9) to prove short-time upper bounds for the average purity of a local region  $A$ . For simplicity, we will consider a uniform distribution  $q$  over  $\mathcal{L}$ .

From Eq. (9) it follows that the maps  $\mathcal{R}$  act on  $\mathcal{CT}_\Lambda$  according the rule  $\mathcal{R}(T_A) = \sum_{B \subset \Lambda} R_{B,A} T_B$ , ( $A \subset \Lambda$ ) where

$$R_{B,A} = \sum_{\Omega \in \mathcal{L} \cap (\partial A)^c} q(\Omega) \delta_{B,A} + \sum_{\Omega \in \partial A} q(\Omega) (\alpha_+(A, \Omega) \delta_{B,A-\Omega} + \alpha_-(A, \Omega) \delta_{B,A \cup \Omega}). \quad (12)$$

For a totally factorized initial state, one has  $P_k = \langle \omega^{\otimes 2}, \mathcal{R}^k(T_A) \rangle = \sum_{B_1, \dots, B_k} R_{B_k, B_{k-1}} R_{B_{k-1}, B_{k-2}} \dots R_{B_1, A}$ . If  $C := |\Omega \cap A| = |\Omega - A|$  then  $c^- = 0 \Rightarrow \alpha_+ = \alpha_- = c^+/2 = d^C/(d^{2C} + 1)$ . In the rest of the section we will assume that  $\alpha_- = \alpha_+ =: N_d$ , ( $\forall \Omega \in \mathcal{L}, A \subset \Lambda$ ) [25]. The matrix (12) can be written as a sum of a diagonal and a off-diagonal matrix  $R = F_0 + F_1$  where  $F_0 := \text{diag}(1 - p(A))_{A \subset \Lambda}$ , ( $p(A) := \sum_{\Omega \in \partial A} q(\Omega) = 1 - \sum_{\Omega \in \mathcal{L} \cap (\partial A)^c} q(\Omega)$ ) and  $(F_1)_{B,A} = N_d \sum_{\Omega \in \partial A} q(\Omega) (\delta_{B,A-\Omega} + \delta_{B,A \cup \Omega})$ . The off-diagonal matrix  $F_1$  connects, with strength  $N_d$ , each region  $A$  with other  $2|\partial A|$ . For example, for  $k = 1$  one immediately finds

$$P_1 = \sum_{B \subset \Lambda} R_{B,A} = \sum_{B \subset \Lambda} ((F_0)_{B,A} + (F_1)_{B,A}) = 1 - p(A) + 2N_d \sum_{\Omega \in \partial A} q(\Omega) = 1 - p(A)(1 - 2N_d),$$

For a uniform distribution  $q$  over  $\mathcal{L}$  one has  $p(A) = |\partial A|/|\mathcal{L}|$ . In general:  $P_k = \sum_{B \subset \Lambda} \left( \sum_{\alpha_1, \dots, \alpha_k=0}^1 F_{\alpha_1} \dots F_{\alpha_k} \right)_{B,A} =: \sum_{[\alpha] \in \mathbf{Z}_2^k} Y_A^{(k)}(\alpha)$ . Using the matrix structure (12), one obtains  $Y_A^{(k)}(\alpha) = \sum_{B \subset \Lambda} (F_{\alpha_1} \dots F_{\alpha_k})_{B,A} \leq (2N_d p(X))^{|\alpha|} (1 - p(\tilde{X}))^{k-|\alpha|}$  where:  $|\alpha| := \sum_{i=1}^k \alpha_i$ ,  $X(\tilde{X})$  is the region in the family obtained by  $A$  with  $k$  successive allowed operations [26] given by either joining or subtracting elements of  $\mathcal{L}$ , with the largest (smallest)  $|\partial \tilde{X}|$  ( $|\partial X|$ ) [27] Whence

$$\begin{aligned} P_k &\leq \sum_{|\alpha|=0}^k C_{k,|\alpha|} (2N_d p(X))^{|\alpha|} (1 - p(\tilde{X}))^{k-|\alpha|} = \left( 1 - p(\tilde{X}) + 2N_d p(X) \right)^k = (1 - (1 - 2N_d)p(X) + p(X) - p(\tilde{X}))^k \\ &= (1 - e_p p(X))^k \left( 1 + \frac{\Delta_{k,A}}{1 - e_p p(X)} \right)^k \leq \exp \left[ -k(p(X)e_p - \frac{\Delta_{k,A}}{1 - e_p}) \right] \end{aligned} \quad (13)$$

where  $C_{k,|\alpha|}$  are binomial coefficients,  $\Delta_{k,A} = p(X) - p(\tilde{X})$  and

$$e_p := 1 - 2N_d = \frac{(d-1)^2}{d^2 + 1}. \quad (14)$$

This constant is the Haar average of the “entangling power” for unitaries acting on  $d \times d$ -dimensional systems introduced in [19]. If we assume that each step the boundary size of a region can change just by a quantity  $O(1) = o(|\partial A|)$  i.e., the hyper-graph  $\mathcal{L}$  has bounded degree and the boundary size of  $A$  is large scaling quantity, it follows  $\Delta_k = O(k/|\mathcal{L}|)$  whereas  $p(X) = O(|\partial A|/|\mathcal{L}|)$ . In words: inequality (13) describes, for  $k = O(1)$ , an exponentially bounded decay of average purity with a rate that fulfills an *area law* (up to corrections  $O(1)$ ) and that is proportional to the average entangling power  $e_p$  [9].



### C. An exactly solvable model: the uniform path-graph

In this section we will discuss a one-dimensional uncorrelated case where the use of swap algebra formalism allows for an explicit exponential reduction of computational complexity [28]. The ensemble map  $\mathcal{R}$  can be explicitly diagonalized in the relevant invariant sub-space of  $\mathbf{CT}_\Lambda$  and bounds on the convergence rate  $P_k \rightarrow P_\infty$  are easily established.

Let  $\mathcal{G} = (\Lambda, E)$  be a *path-graph* of length  $L : \Lambda := \{1, \dots, L\}$ ,  $E = \{\{1, 2\}, \{2, 3\}, \dots, \{L-1, L\}\}$ . From Eq (9) it follows that the  $(L+1)$ -dimensional subspace of  $\mathbf{CT}_\Lambda$  spanned by  $\{T_\Omega / \Omega = \{1, \dots, i\}, i = 1, \dots, L\}$  and  $T_\emptyset = \mathbb{1}$  is invariant under all the edge maps  $\mathcal{R}_e (e \in E)$ . We denote the basis of this  $(L+1)$ -dimensional invariant subspace by  $\{|0\rangle, |1\rangle, |2\rangle, \dots, |L\rangle\}$ . In the uniform case one has  $\mathcal{R} = \frac{1}{L-1} \sum_{i=1}^{L-1} \mathcal{R}_{\{i, i+1\}}$  and relations (12) give:

$$\mathcal{R}|0\rangle = |0\rangle, \mathcal{R}|L\rangle = |L\rangle, \quad \mathcal{R}|i\rangle = a|i\rangle + b(|i-1\rangle + |i+1\rangle), (i = 1, \dots, L-1) \quad (15)$$

where  $a := (L-2)/(L-1) = 1 - (L-1)^{-1}$  and  $b = N_d(L-1)^{-1}$ . Let us denote by  $R$  the (non-hermitean) matrix representation of  $\mathcal{R}$  in the basis  $\{|i\rangle\}_{i=0}^L$ . Eq (15) implies that  $R$  is a sum of two matrices i.e.,  $R = R_1 + R_2$  where  $R_1 = |0\rangle\langle 0| + b|0\rangle\langle 1| + |L\rangle\langle L| + b|L\rangle\langle L-1|$  and  $R_2 = a\mathbb{1}_{L-1} + bA_{L-1}$ . Here  $\mathbb{1}_{L-1}$  denotes the identity matrix in the  $(L-1)$ -dimensional space spanned by  $\{|i\rangle\}_{i=1}^{L-1}$  and  $A_{L-1}$  is the *adjacency* matrix of the path graph of length  $L-1$  whose nodes are labelled by the same basis vectors. The matrix  $R$  is diagonalizable by a (non-unitary) transformation  $R = P^{-1}DP$  where  $D = \text{diag}(\lambda_\mu)_{\mu=0}^L$  and  $P$  is a matrix whose columns are normalized eigenvectors of  $R$ , i.e.,  $P_{j,\mu} = \langle j|\Psi_\mu\rangle$ ,  $R|\Psi_\mu\rangle = \lambda_\mu|\Psi_\mu\rangle$ ,  $(j, \mu = 0, \dots, L)$  [29]. The spectrum of  $R$  is:  $\{\lambda_\mu\}_\mu := \{1, a + 2b \cos(\frac{\pi h}{L}), (h = 1, \dots, L-1), 1\}$ . The  $L-1$  eigenvalues different from one can be written as  $1 - \Delta_h$  where  $\Delta_h := \frac{1}{L-1}(1 - 2N_d \cos(\frac{\pi h}{L})) \in (0, 1)$ . The *spectral gap*  $\Delta := 1 - \max\{\lambda_h\}_{h=1}^{L-1}$  is then given by

$$\Delta = \min_h \Delta_h = \frac{1}{L-1} \left(1 - 2N_d \cos\left(\frac{\pi}{L}\right)\right) \geq \frac{1}{L}(1 - 2N_d) = \frac{e_p}{L}. \quad (16)$$

Here  $e_p$  is the average entangling power defined in Eq. (14). The lower bound (16) shows that the spectral gap of the ensemble map  $\mathcal{R}$  of the uniform path graph is “large” in the sense that it does not vanish faster than  $L^{-1}$  for large system size  $L$ . This fact, in turn implies a fast convergence rate  $P_k \mapsto P_\infty$  for  $k \rightarrow \infty$ .

For general  $k$  and initial state  $|l\rangle$  one has  $P_k = \sum_{j=0}^L (R^k)_{j,l} = \sum_{j=0}^L (PDP^{-1})_{j,l}^k = \sum_{j=0}^L (PD^kP^{-1})_{j,l} = \sum_{\mu=0}^L \lambda_\mu^k (\sum_{j=0}^L P_{j,\mu})(P^{-1})_{\mu,l}$ . Since  $\lambda_0 = \lambda_L = 1$  and  $|\lambda_\mu| < 1$  for  $\mu = 1, \dots, L-1$ , one obtains  $P_k - P_\infty = \sum_{h=1}^{L-1} (1 - \Delta_h)^k (\sum_{j=0}^L P_{j,h})(P^{-1})_{h,l}$ . A direct computation shows that

$$P_k - P_\infty = \frac{2}{L} \sum_{h \text{ odd}} (1 - \Delta_h)^k \sin\left(\frac{\pi l h}{L}\right) \left[ \frac{2N_d \sin(\frac{\pi h}{L})}{2N_d \cos(\frac{\pi h}{L}) - 1} + \cot\left(\frac{\pi h}{2L}\right) \right] \quad (17)$$

where the sum is over the odd numbers  $h$  between 1 and  $L-1$ , and, consistently with (11),  $P_\infty = (d^{L-l} + d^l)(d^L + 1)^{-1}$  [20]. From this equation and the spectral gap definition (16) a straightforward argument shows that  $|P_k - P_\infty| \leq l e^{-k\Delta} C$  where  $C = O(1)$  is a constant depending on  $d$ . Then, using the spectral gap lower bound in (16), one obtains the convergence rate estimate

$$k \geq \frac{L}{e_p} (\log(\frac{C}{\epsilon}) + \log l) \Rightarrow |P_k - P_\infty| \leq \epsilon. \quad (18)$$

This inequality shows that, for fixed accuracy  $\epsilon$ , the circuit length  $k$  scales *linearly* with the total system size [11]. Interestingly (18) also shows that the convergence rate is proportional to the average entangling power  $e_p$ , an intuitive result given the physical meaning of this quantity [19].

We would like to end this section by showing that in this one-dimensional case the bound (13) (with  $\Delta_{k,A} = 0$ ) is tight for short times  $k$ . To begin with, notice that if  $k \leq \min\{l, L-l\}$  the action of the map  $\mathcal{R}^k$  on  $|l\rangle$  does not involve the boundary states  $|0\rangle$  and  $|L\rangle$ . As long as these latter fixed states can be discarded Eq. (15) shows that  $\mathcal{R}$  acts as the sum of three commuting translation operators i.e.,  $\mathcal{R} \cong \sum_{\alpha=0,\pm 1} c_\alpha T_\alpha$  where  $T_\alpha|l\rangle = |l+\alpha\rangle$ ,  $(l = 1, \dots, L-1; \alpha = 0, \pm 1)$  and  $c_0 = a$ ,  $c_{\pm 1} = b$ . Therefore  $\mathcal{R}^k|l\rangle = \sum_{\alpha_1, \dots, \alpha_k=0,\pm 1} c_{\alpha_1} \cdots c_{\alpha_k} T_{\alpha_1 + \dots + \alpha_k}|l\rangle$ . As remarked in the above, for a totally factorized initial state, each vector in the last formula gives a unit contribution to the average purity  $P_k$ , hence  $P_k = \sum_{\alpha_1, \dots, \alpha_k=0,\pm 1} c_{\alpha_1} \cdots c_{\alpha_k} = (c_0 + c_{-1} + c_1)^k = (a + 2b)^k = (1 - \frac{1}{L-1}e_p)^k$ . This simple result can be also checked by resorting to the explicit formula (17).

### D. Correlated case

In this subsection we will consider an ensemble map of the form  $\mathcal{R} = \prod_{\Omega \in \mathcal{L}} \mathcal{R}_\Omega$ . This is a natural generalization of the correlated family of 1-dimensional L-RQC called the “contiguous edge model” introduced and analyzed in [9, 10]. In the contiguous edge model the graph vertex set is given by  $\Lambda = \{-L, -L+1, \dots, -1, 0, 1, \dots, L\}$  and the edge set  $E$  as in Sect. C. The L-RQC model is now defined as follows: At each discrete time  $t = k$  one selects  $2L$  (Haar) random  $U_e$  and  $\mathbf{U}$  is built according to a permutation  $\pi \in \mathcal{S}_{|E|}$ , ( $|E| = 2L$ )  $\mathbf{U} = U_{e_{\pi(1)}} U_{e_{\pi(2)}} \dots U_{e_{\pi(|E|)}}$  and then the process is iterated at  $t = k+1$  and so on. Each  $U_{e_j}$  is unitary acting on the edge  $e_j = (j, j+1)$  state space  $h_j \otimes h_{j+1} \cong (\mathbf{C}^d)^{\otimes 2}$ . The ensemble map corresponding to the specified model is given by  $\mathcal{R}_\pi := \mathcal{R}_{e_{\pi(|E|)}} \circ \dots \circ \mathcal{R}_{e_{\pi(1)}}$ . Concerning the map fixed point, in [10] it was conjectured that for *all*  $\pi \in \mathcal{S}_{|E|}$  the relevant fixed point of  $\mathcal{R}_\pi$  is given by  $\mathbb{1} + T_\Lambda$  and  $P_\infty$  by (11). In the following we will demonstrate a result (Prop. 5) that, as a particular case, proves these conjectures.

Notice that, since in general different  $\mathcal{R}_\Omega$  do not commute, the map  $\mathcal{R} = \prod_{\Omega \in \mathcal{L}} \mathcal{R}_\Omega$  it is not even necessarily Hermitean. Therefore one needs a different type of analysis respect to the uncorrelated case of Prop. 4.

#### Lemma

Let  $P_1, \dots, P_n$  be projectors in a finite-dimensional Hilbert Space,  $R =: P_n P_{n-1} \dots P_1$  and  $F := \cap_{i=1}^n \text{Im } P_i$ . Then **i)**  $\text{Fix}(R) = F$ , **ii)**  $\|R\| = 1 \Leftrightarrow \text{Fix}(R) \neq \{0\}$ , **iii)**  $R = P + Q$  where  $P$  is the projection on  $F$  and  $\|Q\| < 1$ .

**Proof.**— **i)** Obviously  $F \subset \text{Fix}(R)$ , let us show the reverse inclusion. Let  $x_0 \in \text{Fix}(R)$  and  $x_i = P_i x_{i-1}$ , ( $i = 1, \dots, n$ ). One has  $x_0 = R(x_0) = x_n = P_n(x_{n-1})$  and then  $\|x_0\| = \|P_n x_{n-1}\| \leq \|P_n\| \|x_{n-1}\| \leq \|P_n\| \|P_{n-1}\| \|x_{n-2}\| \leq \dots \leq \prod_{i=1}^n \|P_i\| \|x_0\| \leq \|x_0\|$ , this shows that the equalities  $\|P_i x_{i-1}\| = \|x_{i-1}\|$ , ( $i = 1, \dots, n$ ) hold. These latter, since the  $P_i$ 's are projections, imply  $x_i := P_i x_{i-1} = x_{i-1}$  ( $i = 1, \dots, n$ ). Therefore  $x_n = x_{n-1} = \dots = x_1 = x_0$ , namely  $P_i x_0 = x_0$  for all  $i$  i.e.,  $x_0 \in F$ .

**ii)** If  $F \neq \{0\}$  obviously  $\|R\| = 1$ ; let us show the reverse implication. If  $\|R\| = 1$  from compactness of the unit sphere  $\exists x^*, \|x^*\| = 1$  s.t.  $\|R x^*\| = \|x^*\|$ . From this relation, following the same steps as in i), one sees that  $P_i(x^*) = x^*$  for all  $i = 1, \dots, n$  i.e.,  $0 \neq x^* \in F$ . Since  $\|R\| \leq \prod_{i=1}^n \|P_i\| = 1$ , if  $F = \{0\}$  then  $\|R\| < 1$ .

**iii)**  $F$  is a subspace of all the spaces  $\text{Im } P_i$ , ( $i = 1, \dots, n$ ) hence one can write  $P_i = P + Q_i$  where  $Q_i$  projects on the orthogonal complement (in  $\text{Im } P_i$ ) of  $F$ . Now we have  $R = (P + Q_n)(P + Q_{n-1}) \dots (P + Q_1) = P + Q_n Q_{n-1} \dots Q_1 =: P + Q$  where we have used  $Q_i P = P Q_i = 0$ , ( $i = 1, \dots, n$ ). Notice now that  $Q = R(1 - P)$  therefore  $Q$  is a product of projections that (by construction) cannot have a non-trivial joint fixed eigenspace. From ii) it follows that  $\exists \Delta \in (0, 1)$  s.t.  $\|Q\| = 1 - \Delta$ .  $\square$

#### Proposition 5

**i)** Let  $\mathcal{R} = \mathcal{R}_{\Omega_{|\mathcal{L}|}} \mathcal{R}_{\Omega_{|\mathcal{L}|-1}} \dots \mathcal{R}_{\Omega_1}$ , ( $\{\Omega_i\}_{i=1}^{|\mathcal{L}|} := \mathcal{L}$ ) be the ensemble map of a correlated L-RQC. For a pure totally factorized initial state  $\omega$  one has

$$P_k = \langle \omega^{\otimes 2}, \mathcal{R}^k(T_\Omega) \rangle = P_\infty + O(2^{|\Lambda|/2} e^{-Ak}), \quad A := \log \frac{1}{1 - \Delta}, \quad (19)$$

where  $P_\infty$  is given by (11) and  $1 - \Delta = \|\mathcal{R}(1 - \mathcal{P})\| < 1$ ,  $\mathcal{P}$  projector on  $\text{Fix}(\mathcal{R})$  (see Prop. 4).

**ii)** If  $\mathcal{R}_\pi := \mathcal{R}_{\Omega_{\pi(|\mathcal{L}|)}}$  has the same infinite time purity Eq. (19)  $\forall \pi \in \mathcal{S}_{|\mathcal{L}|}$ .

**Proof.**— **i)** We are in the setting of the Lemma with  $F \neq \{0\}$  (Prop. 4) where now the finite-dimensional Hilbert space is the Hilbert-Schmidt space  $\mathcal{L}(\Lambda)^{\otimes 2}$  the projections are the  $\mathcal{R}_\Omega$  and  $n = |\mathcal{L}|$ .

One has  $\mathcal{R}^k = (\mathcal{P} + \mathcal{Q})^k = \mathcal{P} + \mathcal{Q}^k$  ( $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0$ ). Since  $\|\mathcal{Q}^k\| \leq \|\mathcal{Q}\|^k = (1 - \Delta)^k = \exp\left(-k \log \frac{1}{1 - \Delta}\right)$  it follows that  $\lim_{k \rightarrow \infty} \mathcal{Q}^k = 0$  and hence  $\lim_{k \rightarrow \infty} \mathcal{R}^k = \mathcal{P}$ . Therefore  $P_k = \langle \omega^{\otimes 2}, (\mathcal{P} + \mathcal{Q}^k)(T_\Omega) \rangle = P_\infty + \langle \omega^{\otimes 2}, \mathcal{Q}^k(T_\Omega) \rangle$ , where  $P_\infty = \langle \omega^{\otimes 2}, \mathcal{P}(T_\Omega) \rangle$  is given (see Prop. 0) by Eq. (11). Now, using the Remark after Prop. 4, and Schwarz inequality for the Hilbert-Schmidt scalar product we can write  $\langle \omega^{\otimes 2}, \mathcal{Q}^k(T_\Omega) \rangle = d_+^{-|\Lambda|} \langle \Pi^+, \mathcal{Q}^k(T_\Omega) \rangle \leq d_+^{-|\Lambda|} \|\Pi^+\|_2 \|\mathcal{Q}^k(T_\Omega)\|_2 \leq d_+^{-|\Lambda|/2} \|\mathcal{Q}^k(T_\Omega)\|_2 \leq d_+^{-|\Lambda|/2} \|\mathcal{Q}^k\| \|T_\Omega\|_2 \leq \left(\frac{2d}{d+1}\right)^{|\Lambda|/2} \|\mathcal{Q}\|^k \leq \left(\frac{2d}{d+1}\right)^{|\Lambda|/2} (1 - \Delta)^k \leq 2^{|\Lambda|/2} (1 - \Delta)^k$ . Where we also used  $\|T_\Omega\|_2^2 = \text{Tr } T_\Omega^2 = \text{Tr } \mathbb{1}_\Lambda = d^{2|\Lambda|}$ . This proves (19).

**ii)** Point i) shows that infinite time purity  $P_\infty$  depends just on the joint fixed space  $\cap_{i=1}^{|\mathcal{L}|} \text{Im } \mathcal{R}_{\pi(i)} = \text{Im } \mathcal{P}$ . This latter is clearly independent on the permutation  $\pi$ .  $\square$

In the decomposition  $\mathcal{R}_\pi = \mathcal{P} + \mathcal{Q}_\pi$  the second term *does* in general depend on  $\pi$  Accordingly the convergence rate of  $P_k$  to  $P_\infty$  (related, by Prop. 6, to  $\|\mathcal{Q}_\pi\|$ ) may depend on  $\pi$ . This phenomenon has been numerically observed in [9, 10]. From (19) it follows that  $k \gg \Delta^{-1} \log(2^{|\Lambda|/2}/\epsilon) \Rightarrow |P_k - P_\infty| = O(\epsilon)$ . This shows how the convergence rate  $P_k \rightarrow P_\infty$  is controlled by the norm  $\|\mathcal{Q}\| = 1 - \Delta$ . This norm is also the second largest singular value of  $\mathcal{R}$ . The

positive constant  $\Delta$  is therefore just the difference between the two largest singular values of the ensemble map  $\mathcal{R}$  and it is often referred to as the spectral gap (see Sect C). From the practical point of view it is crucial to determine how the convergence varies with the type of L-RQC and how the spectral gap scales as function of  $|\Lambda|$  [11, 12]. In [9] and [10] are given numerical results for a few natural choices of  $\pi$ . In particular for the so called expanding  $\pi$  (see Sect V.A in [10]), arguably the less efficient at generating entanglement, it was found  $P_k = 2(1 - e_p)^k(1 + e_p)^{-k}$  ( $e_p$  is defined in (14) for  $1 \ll k \leq |\Omega|$  and a cross over to a volume-law for  $k = |\Omega|$ ). Moreover: 1) asymptotic purity is given by (11) (for  $K = 1$ ) and, 2) the spectral gap of  $\mathcal{R}_{\pi}$ , was found (numerically) to have a finite limit,  $1 - (2N_d)^2$ , for  $L \rightarrow \infty$ .

We end this section by noticing that, for both the uncorrelated and correlated cases, Propositions 4 and 5 show that the fixed point problem for the ensemble maps  $\mathcal{R}$  can be formulated as a ground state problem for the ensemble “Hamiltonians”  $\mathcal{K} := \sum_{\Omega \in \mathcal{L}} (\mathbf{1} - \mathcal{R}_{\Omega})$ . This are *local* non-negative operators acting on the Hilbert-Schmidt space  $\mathcal{A}(\Lambda)^{\otimes 2} \cong \otimes_{i \in \Lambda} \mathcal{B}(h_i^{\otimes 2})$ . In fact one has  $\mathcal{K}\Phi = 0 \Leftrightarrow \mathcal{R}_{\Omega}\Phi = \Phi, \forall \Omega \in \mathcal{L}$ , namely the ground state manifold of  $\mathcal{K}$  coincides with the  $2^K$ -dimensional space  $\text{Fix}(\mathcal{R})$ . It would then interesting to see whether one may adapt to the present context the techniques developed in [8, 21] to lower bound spectral gap of local Hamiltonians as well as to see whether one may exploit in convergence rate estimates the so-called quantum detectability lemma [22].

## VI. ASYMPTOTIC STATISTICS AND LOCAL STATE DESIGNS

In this section we will establish a simple connection between the formalism and the results of this paper and the theory of quantum local  $t$ -designs [7, 8, 12].

The asymptotic average purity value (11) is  $O(d^{-|\Omega^c|})$  away from the minimal possible one for the reduced state of the region  $\Omega$  i.e.,  $P_{\min} = d^{-|\Omega|}$ . This implies that for sufficiently large  $k$  most of the states  $\omega_{\Omega}(\mathbf{U}_k) := \text{tr}_{\Omega^c}(\mathbf{U}_k \omega \mathbf{U}_k^{\dagger})$  ( $\mathbf{U}_k$  RQC of length  $k$ ) should be close to the maximally mixed state  $\mathbf{1}_{\Omega}/d^{|\Omega|}$ . Indeed one can prove that if  $k$  is such that  $|P_k - P_{\infty}| \leq \epsilon$  then [10]

$$\overline{\|\omega_{\Omega}(\mathbf{U}_k) - \frac{\mathbf{1}_{\Omega}}{d^{|\Omega|}}\|_1}^{\mathbf{U}_k} \leq \sqrt{d^{|\Omega|}} \sqrt{\frac{1}{d^{|\Omega^c|}}} + \epsilon. \quad (20)$$

Therefore if  $|\Omega^c| = (1 + \alpha)|\Omega|$  ( $\alpha > 0$ ) and  $\epsilon = d^{-|\Omega^c|}$  one has that the average trace norm distance between states generated by the ensemble of RQCs and the maximally mixed one is  $O(d^{-\alpha|\Omega|/2})$ . Using Markov inequality, for large  $|\Omega|$ , one sees that with high probability each state in the ensemble is *locally nearly indistinguishable from the totally mixed one*. Notice now that similar claims can be made for the Haar measure as:  $\overline{\text{tr} \omega_{\Omega}(\psi)^2}^{\psi} \leq d_{|\Omega|}^{-1} + d_{|\Omega^c|}^{-1}$  [19] where  $\omega_{\Omega}(\psi) := \text{tr}_{\Omega^c} |\psi\rangle\langle\psi|$ .

All this suggests that statistics of *local* observables in  $\Omega$  over the L-RQC ensemble (for large  $k$ ) should be a good approximation of the Haar’s one. In order to see this one has to show that the distance  $\mathcal{D}_t := \|\overline{\omega_{\Omega}(\mathbf{U}_k)^{\otimes t}}^{\mathbf{U}_k} - \overline{\omega_{\Omega}(\psi)^{\otimes t}}^{\psi}\|_1$ , ( $t$ -positive integer) is small e.g., less than  $\delta$  in some limit. If this is the case all local observables will have all the L-RQC statistical moments up to the  $t$ -th one at most  $\delta$  away from the corresponding Haar’s one [7]. An ensemble of states having this property i.e., to provide a way to approximately sample the Haar distribution is called an  $\delta$ -approximate *state  $t$ -design*. How this can be done by efficiently resorting just to local quantum circuits is a problem currently at the centre of an intense attention in view of its applications to quantum information theory (see recent [8]).

**Proposition 6** For  $k = O(\Delta^{-1} \log d(|\Lambda| + (1 + \alpha)|\Omega|))$ , ( $\alpha > 0$ ) the L-RQC family is a  $\delta$ -approximate state  $t$ -design for local observables  $A \in \mathcal{A}(\Omega)$  with  $\delta = O(\sqrt{te^{-\alpha|\Omega|}})$ .

**Proof.**— For  $k = O(\Delta^{-1} \log d(|\Lambda| + (1 + \alpha)|\Omega|))$  bound (??) implies  $|P_k - P_{\infty}| \leq d^{-(1+\alpha)|\Omega|} := \epsilon$  Using triangular inequality, convexity of the trace norm, Eq. (20) and  $\|\omega^{\otimes t} - \sigma^{\otimes t}\|_1 \leq \sqrt{t}\|\omega - \sigma\|_1$  (for all pairs of density matrices  $\rho$  and  $\sigma$ ) one obtains  $\mathcal{D}_t \leq \|\overline{\omega_{\Omega}(\mathbf{U}_k)^{\otimes t} - \frac{\mathbf{1}_{\Omega}}{d^{|\Omega|}}^{\otimes t}}^{\mathbf{U}_k}\|_1 + \|\overline{\frac{\mathbf{1}_{\Omega}}{d^{|\Omega|}}^{\otimes t} - \omega_{\Omega}(\psi)^{\otimes t}}^{\psi}\|_1 \leq \|\overline{\omega_{\Omega}(\mathbf{U}_k)^{\otimes t} - \frac{\mathbf{1}_{\Omega}}{d^{|\Omega|}}^{\otimes t}}^{\mathbf{U}_k}\|_1 + \|\overline{\frac{\mathbf{1}_{\Omega}}{d^{|\Omega|}}^{\otimes t} - \omega_{\Omega}(\psi)^{\otimes t}}^{\psi}\|_1 \leq \sqrt{t} \left( \sqrt{\frac{d^{|\Omega|}}{d^{|\Omega^c|}}} + d^{|\Omega|} \epsilon + \sqrt{\frac{d^{|\Omega|}}{d^{|\Omega^c|}}} \right) =: \delta = O(\sqrt{te^{-\alpha|\Omega|}})$   $\square$ .

## VII. CONCLUSIONS

In this paper we provided a mathematical presentation of the approach to local random circuits discussed in [9, 10]. We introduced different classes of local random quantum circuits and showed that their statistical properties are encoded into an associated family of completely positive maps. Ensemble average of quantum expectation can be studied as a function of the circuit length and we proved infinite time results as well as convergence bounds. Remarkably, average entanglement dynamics can be described by the action of ensemble maps on operator algebras of permutations (swap algebras). Also in this case we proved infinite time results for the expectation value of the purity of a local region both for uncorrelated and correlated local random quantum circuits and short time area-law bounds for the uncorrelated case. The swap algebra formalism is powerful as it allows in some case an exponential reduction of the problem size i.e.,  $d^{2L} \mapsto L$ . We illustrated such a phenomenon by an exactly solvable one dimensional uncorrelated case. Finally we briefly discussed a connection with  $t$ -design [7, 8] for localized observables.

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- [1] R. Goodman and N. R. Wallach, *Representations and Invariants of Classical Groups*, Encyclopedia of Mathematics and its Applications, Volume 68
  - [2] J. Emerson, Y.S. Weinstein, M. Saraceno, S. Lloyd, and D.G. Cory, *Pseudo-Random Unitary Operators for Quantum Information Processing*, Science **302**, 2098 (2003)
  - [3] D. Poulin, A. Qarry, R. D. Somma, and F. Verstraete, *Quantum simulation of time-dependent Hamiltonians and the convenient illusion of Hilbert space*, Phys. Rev. Lett. **106**, , 170501(2011)
  - [4] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zangh, *Canonical Typicality* , Phys. Rev. Lett. **96**, 050403 (2006).
  - [5] S. Popescu, A. J. Short, and A. Winter, *Entanglement and the foundations of statistical mechanics* , Nat. Phys. **2**, 754 (2006).
  - [6] S. Garnerone, T. R. de Oliveira and P. Zanardi, *Typicality in random matrix product states*, Phys. Rev. A **81**, 032336 (2010); S. Garnerone, T. R. de Oliveira, S. Haas and P. Zanardi, *Statistical properties of random matrix product states*, Phys. Rev. A **82**, 052312 (2010)
  - [7] A. W. Harrow, R. A. Low, *Random Quantum Circuits are Approximate 2-designs*, Comm. Math. Phys. Vol. **291**, No. 1, pp. 257–302 (2009)
  - [8] F.G. S. L. Brandao, A. W. Harrow, M. Horodecki, *Local random quantum circuits are approximate polynomial-designs*, arXiv:1208.0692
  - [9] A.Hamma, S. Santra and P. Zanardi, *Quantum Entanglement in Random Physical States*, Phys. Rev. Lett. **109**, 040502 (2012)
  - [10] A.Hamma, S. Santra and P. Zanardi, *Ensembles of physical states and random quantum circuits on graphs*, Phys. Rev. A **86**, 052324 (2012)
  - [11] M. Znidaric, *Exact convergence times for generation of random bipartite entanglement*, Phys. Rev. A **78**, 032324 (2008)
  - [12] W.G. Brown, L. Viola, *Exact convergence rates for arbitrary statistical moments of random quantum circuits*, Phys. Rev. Lett. **104**, 250501 (2010)
  - [13] O.C.O. Dahlsten, R. Oliveira, M.B. Plenio, *Efficient generation of generic entanglement*, Phys. Rev. Lett. **98**, 130502 (2007).
  - [14] N. Linden, S. Popescu, A. J. Short, and A. Winter, *Quantum mechanical evolution towards thermal equilibrium*, Phys. Rev. E **79**, 061103 (2009)
  - [15] D.V. Voiculescu, K.J. Dykema, and A. Nica. *Free Random Variables*, volume 1 of CRM Monograph Series. American Mathematical Society, 1992.
  - [16] K. Kraus, States, Effects and Operations: *Fundamental Notions of Quantum Theory*, Springer Verlag 1983
  - [17] C. H. Bennett and D. P. DiVincenzo, *Quantum information and computation*, Nature **404**, 247-255 (2000)
  - [18] M. B. Hastings, I. Gonzalez, A. B. Kallin, and R. G. Melko, *Measuring Renyi Entanglement Entropy in Quantum Monte Carlo Simulations*, Phys. Rev. Lett. **104**, 157201;
  - [19] P. Zanardi, Ch. Zalka and L. Faoro, *Entangling power of quantum evolutions*. Phys. Rev. A. **62**, 030301(2000)
  - [20] J. Kaniewski, private communication
  - [21] B. Nachtergaele, *The spectral gap for some spin chains with discrete symmetry breaking*, Commun. Math. Phys., 175 (1996) 565-606
  - [22] D. Aharonov, I. Arad, U. Vazirani, and Z. Landau *The detectability lemma and its applications to quantum hamiltonian complexity*, New Journal of Physics, 13(11):113043, 2011.

- [23] For example:  $\Omega_1 \subset \Omega_2 \Rightarrow \text{Im } \mathcal{R}_{\Omega_1} \supset \text{Im } \mathcal{R}_{\Omega_2} \Rightarrow \mathcal{R}_{\Omega_2}(1 - \mathcal{R}_{\Omega_1}) = 0$ .
- [24] A connected component in  $\mathcal{L}$  is a maximal set  $C_i \subset \cup_{\Omega \in \mathcal{L}} \Omega \subset \Lambda$  such that  $\forall u, v \in C_i, \exists \Omega_1, \dots, \Omega_n \in \mathcal{L}$  such that  $u \in \Omega_1, v \in \Omega_n$  and  $\Omega_i \cap \Omega_{i+1} \neq \emptyset, (i = 1, \dots, n-1)$  i.e., given any two vertices  $u$  and  $v$  there is a path of hyperlinks connecting them.
- [25] This is the case when the elements  $\Omega_1$  in  $\mathcal{L}$  are edges of a graph  $\Omega_1 = \{v_1, v_2\} (v_1, v_2 \in \Lambda)$ . In fact  $\Omega_1 \in \partial\Omega \Rightarrow \Omega_1 - \Omega = \{v_1\}$ , and  $\Omega_1 \cap \Omega = \{v_2\}$ . Here  $N_d = d/(d^2 + 1)[9]$ .
- [26] By allowed we mean that at each stage the element  $\Omega \in \mathcal{L}$  added or subtracted to  $A'$  has to belong to  $\partial A'$ .
- [27] Notice that  $X$  and  $\tilde{X}$  may depend on both  $A$  and  $k$ .
- [28] In [10] it was analyzed the case of the complete graph with  $L$  vertices. Also in this case one has an exponential reduction of complexity as average purity dynamics can be restricted to the  $(L+1)$ -dimensional  $\mathcal{S}_L$ -symmetric subspace of  $\mathcal{CT}_\Lambda \cong (\mathcal{C}^2)^{\otimes L}$ .
- [29] Explicitly one finds:  $|\Psi_0\rangle = |0\rangle, |\Psi_L\rangle = |L\rangle$  and  $|\Psi_h\rangle = \mathcal{N}_h \left( \sum_{j=1}^{L-1} c_h(j)|j\rangle - b/\Delta_h (c_h(1)|0\rangle + c_h(L-1)|L\rangle) \right)$ , where  $c_h(j) := \sin(\pi h j/L)$  for  $h = 1, \dots, L-1$ . The normalization constant  $\mathcal{N}_h$  is given by  $\mathcal{N}_h = (L/2 + (b/\Delta_h)^2(c_h^2(1) + c_h(L-1)^2))^{-1/2} = O(L^{-1/2})$ .

### Appendix A: Fixed swaps subalgebras

In this section we will show how some of the facts used in the Proof. of Proposition. 4 can be established working entirely within the swap algebra formalism.

#### Proposition A1

Let  $\mathcal{R}_{\Omega_1}$  be an ensemble map given by (9),  $\Omega_1, \Omega_2 \subset \Lambda$  and  $\text{Fix}(\mathcal{R}_{\Omega_1}) := \text{span}\{T_\Omega / \mathcal{R}_{\Omega_1}(T_\Omega) = T_\Omega\}$ .

- a)  $\Lambda_1 \subset \Lambda$  implies that  $\mathcal{CT}_{\Lambda_1} = \text{span}\{T_\Omega / \Omega \subset \Lambda_1 \subset \Lambda\}$  is a subalgebra of  $\mathcal{CT}_\Lambda$  of dimension  $2^{|\Lambda_1|}$ .
- b) If  $\Lambda_1 \cap \Lambda_2 = \emptyset \Rightarrow \mathcal{CT}_{\Lambda_1 \cup \Lambda_2} \cong \mathcal{CT}_{\Lambda_1} \vee \mathcal{CT}_{\Lambda_2} \cong \mathcal{CT}_{\Lambda_1} \otimes \mathcal{CT}_{\Lambda_2}$ .
- c)  $\text{Fix}(\mathcal{R}_{\Omega_1}) = \text{span}\{T_\Omega / \Omega \supset \Omega_1 \vee \Omega \subset \Omega_1^c\} = \text{span}\{T_\Omega / \Omega_1 \notin \partial\Omega\}$ ,  $\dim \text{Fix}(\mathcal{R}_{\Omega_1}) = 2^{|\Lambda| - |\Omega_1| + 1}$
- d)  $\text{Fix}(\mathcal{R}_{\Omega_1})$  is a subalgebra of  $\mathcal{CT}_\Lambda$  isomorphic to  $\mathcal{C}\{\mathbb{1}, T_{\Omega_1}\} \otimes \mathcal{CT}_{\Omega_1^c}$ .
- e)  $\Omega_1 \cap \Omega_2 = \emptyset \Rightarrow \text{Fix}(\mathcal{R}_{\Omega_1}) \cap \text{Fix}(\mathcal{R}_{\Omega_2}) \cong \mathcal{C}\{\mathbb{1}, T_{\Omega_1}\} \otimes \mathcal{C}\{\mathbb{1}, T_{\Omega_2}\} \otimes \mathcal{CT}_{(\Omega_1 \cup \Omega_2)^c}$
- f)  $\Omega_1 \cap \Omega_2 \neq \emptyset \Rightarrow \text{Fix}(\mathcal{R}_{\Omega_1}) \cap \text{Fix}(\mathcal{R}_{\Omega_2}) \cong \text{Fix}(\mathcal{R}_{\Omega_1 \cup \Omega_2}) \cong \mathcal{C}\{\mathbb{1}, T_{\Omega_1 \cup \Omega_2}\} \otimes \mathcal{CT}_{(\Omega_1 \cup \Omega_2)^c}$ .

**Proof.**—

a) Obvious.

b) Any element of  $\Lambda_1 \cup \Lambda_2$ , when  $\Lambda_1 \cap \Lambda_2 = \emptyset$ , can be written in unique way as  $\Omega_1 \cup \Omega_2$  where  $\Omega_i \subset \Lambda_i (i = 1, 2)$ . Therefore  $T_{\Omega_1 \cup \Omega_2} = T_{\Omega_1 \Delta \Omega_2} = T_{\Omega_1} T_{\Omega_2}$  showing the first isomorphism. Moreover, the map  $\iota(T_{\Omega_1 \cup \Omega_2}) = T_{\Omega_1} \otimes T_{\Omega_2}$  can be checked to provide the second one. [Notice also  $\dim \mathcal{CT}_{\Lambda_1 \cup \Lambda_2} = 2^{|\Lambda_1 \cup \Lambda_2|} = 2^{|\Lambda_1|} 2^{|\Lambda_2|} = \dim \mathcal{CT}_{\Lambda_1} \dim \mathcal{CT}_{\Lambda_2}$ .] If  $\Omega_i, \tilde{\Omega}_i \in \Lambda_i (i = 1, 2)$  one has  $(\Omega_1 \cup \Omega_2) \Delta (\tilde{\Omega}_1 \cup \tilde{\Omega}_2) = (\Omega_1 \Delta \tilde{\Omega}_1) \cup (\Omega_2 \Delta \tilde{\Omega}_2)$ ; it follows that  $\iota(T_{\Omega_1 \cup \Omega_2} T_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2}) = \iota(T_{(\Omega_1 \cup \Omega_2) \Delta (\tilde{\Omega}_1 \cup \tilde{\Omega}_2)}) = \iota(T_{(\Omega_1 \Delta \tilde{\Omega}_1) \cup (\Omega_2 \Delta \tilde{\Omega}_2)}) = T_{\Omega_1 \Delta \tilde{\Omega}_1} \otimes T_{\Omega_2 \Delta \tilde{\Omega}_2} = (T_{\Omega_1} \otimes T_{\Omega_2})(T_{\tilde{\Omega}_1} \otimes T_{\tilde{\Omega}_2}) = \iota(T_{\Omega_1 \cup \Omega_2}) \iota(T_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2})$  i.e.,  $\iota$  is also an *algebra* isomorphism (not just a vector space one).

c) Notice first that, in view of (9),  $\text{Fix}(\mathcal{R}_{\Omega_1})$  is a linear subspace of  $\mathcal{CT}_\Lambda$  that contains  $\text{span}\{T_\Omega / \Omega \supset \Omega_1 \vee \Omega \subset \Omega_1^c\}$  (as for every element  $\Omega$  one has  $\Omega_1 \notin \partial\Omega$ ). Moreover this latter subspace has dimension  $2^{|\Lambda| - |\Omega_1| + 1}$  as every element of its basis has the form  $\Omega'$  or  $\Omega' \cup \Omega_1$  where  $\Omega' \subset \Omega_1^c$ . Since  $\mathcal{R}_{\Omega_1}$  is a projector the dimension of  $\text{Fix}(\mathcal{R}_{\Omega_1})$  i.e., the eigenspace of  $\mathcal{R}_{\Omega_1}$  with eigenvalue one, is given by  $\text{Tr } \mathcal{R}_{\Omega_1}$ ; using again (9) one sees that this trace is in fact  $2^{|\Lambda| - |\Omega_1| + 1}$  (only the elements  $\Omega$  such that  $\Omega_1 \notin \partial\Omega$  contribute, each by a one). This proves the equalities in c).

d) If  $\Omega, \Omega'$  are such that  $T_\Omega, T_{\Omega'} \in \text{Fix}(\mathcal{R}_{\Omega_1})$  there are four possibilities 1)  $\Omega \subset \Omega_1^c \wedge \Omega' \subset \Omega_1^c$  2)  $\Omega \supset \Omega_1 \wedge \Omega' \subset \Omega_1^c$ , 3)  $\Omega \subset \Omega_1^c \wedge \Omega' \supset \Omega_1$ , 4)  $\Omega \supset \Omega_1 \wedge \Omega' \supset \Omega_1$ . In all these cases it can be directly checked that  $\Omega \Delta \Omega'$  either contains  $\Omega_1$  or is contained in  $\Omega_1^c$  i.e.,  $\Omega_1 \notin \partial(\Omega \Delta \Omega')$ . Hence  $T_\Omega T_{\Omega'} = T_{\Omega \Delta \Omega'} \in \text{Fix}(\mathcal{R}_{\Omega_1})$ . The isomorphism in d) follows from the fact (noticed in c)) that every element in  $\text{Fix}(\mathcal{R}_{\Omega_1})$  can be written as  $T_{\Omega_1 \cup \Omega'} = \iota^{-1}(T_{\Omega_1} \otimes T_{\Omega'})$  or  $T_{\Omega'} = \iota^{-1}(\mathbb{1} \otimes T_{\Omega'})$ , ( $\Omega' \subset \Omega_1^c$ ).

e) If  $\Omega \in \text{Fix}(\mathcal{R}_{\Omega_1}) \cap \text{Fix}(\mathcal{R}_{\Omega_2})$  (with  $\Omega_1 \cap \Omega_2 = \emptyset$ ) there are four cases: 1)  $\Omega \supset \Omega_1 \wedge \Omega \supset \Omega_2$ , 2)  $\Omega \subset \Omega_1^c \wedge \Omega \supset \Omega_2$ , 3)  $\Omega \supset \Omega_1 \wedge \Omega \subset \Omega_2^c$ , 4)  $\Omega \subset \Omega_1^c \wedge \Omega \subset \Omega_2^c$ . Is easy to see that these alternatives account for the four different types of terms one gets from the tensor product in e) e.g.  $\iota^{-1}(T_{\Omega_1} \otimes T_{\Omega_2} \otimes T_{\Omega'}) = T_{\Omega_1 \cup \Omega_2 \cup \Omega'}$ , ( $\Omega' \subset (\Omega_1 \cup \Omega_2)^c$ ) corresponds to 1) and so on.

f) If  $\Omega_1 \cap \Omega_2 \neq \emptyset$  then cases 2) and 3) in the above are not allowed and one is left with 1) and 4); namely,  $\Omega \supset \Omega_1 \cup \Omega_2$  or  $\Omega \subset \Omega_1^c \cap \Omega_2^c = (\Omega_1 \cup \Omega_2)^c$ . This means  $\Omega_1 \cup \Omega_2 \notin \partial\Omega$  that, thanks to c), is equivalent to say  $T_\Omega \in \text{Fix}(\mathcal{R}_{\Omega_1 \cup \Omega_2})$ . Last isomorphism follows from d).  $\square$